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On inertial waves and oscillations in a rapidly rotating spheroid

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The problem of fluid motions in the form of inertial waves or inertial oscillations in an incompressible viscous fluid contained in a rotating spheroidal cavity was first formulated and studied by Poincaré (1885) and Bryan (1889). Upon realizing the limitation of Bryan's implicit solution using complicated modified spheroidal coordinates, Kudlick (1966) proposed a procedure that may be used to compute an explicit solution in spheroidal coordinates. However, the procedure requires an analytical expression for the N real and distinct roots of a polynomial of degree N, where N is a key parameter in the problem. When $0 \leqslant N \leqslant 2$, an explicit solution can be derived by using Kudlick's procedure. When $3 \leqslant N \leqslant 4$, the procedure cannot be practically used because the analytical expression for the N distinct roots becomes too complicated. When $N > 4$, Kudlick's procedure cannot be used because of the non-existence of an analytical expression for the N distinct roots. For the inertial wave problem, Kudlick thus restricted his analysis to several modes for $1 \leqslant N \leqslant 2$ with the azimuthal wavenumbers $1 \leqslant m \leqslant 2$. We have found the first explicit general analytical solution of this classical problem valid for $0 \leqslant N < \infty$ and $0 \leqslant m < \infty$. The explicit general solution in spheroidal polar coordinates represents a possibly complete set of the inertial modes in an oblate spheroid of arbitrary eccentricity. The problem is solved by a perturbation analysis. In the first approximation, the effect of viscosity on inertial waves or oscillations is neglected and the corresponding inviscid solution, the pressure and the three velocity components in explicit spheroidal coordinates, is obtained. In the next approximation, the effect of viscous dissipation on the inviscid solution is examined. We have derived the first explicit general solution for the viscous spheroidal boundary layer valid for all inertial modes. The boundary-layer flux provides the solvability condition that is required to solve the higher-order interior problem, leading to an explicit general expression for the viscous correction of all inertial modes in a rapidly rotating, general spheroidal cavity. On the basis of the general explicit solution, some unusual and intriguing properties of the spheroidal inertial waves or oscillation are discovered. In particular, we are able to show that

$$\int_V (\boldsymbol{u} \cdot \nabla^2 \boldsymbol{u})\, \mathrm{d}V \equiv 0,$$

where $\boldsymbol{u}$ is the velocity of any three-dimensional inviscid inertial waves or oscillations in an oblate spheroid of arbitrary eccentricity and $\int_V$ denotes three-dimensional integration over the volume of the spheroidal cavity.

1. Introduction

As a consequence of rapid rotation, many astrophysical and planetary fluids are confined in oblate spheroidal cavities in which the flow may be assumed to be inviscid in the first approximation and is then corrected by a thin Ekman boundary layer near the surface of the spheroidal cavity (Stewartson & Roberts 1963). Motivated by geophysical and astrophysical applications, the subject of fluid motions in the form of non-axisymmetric inertial waves or axisymmetric inertial oscillations in a rotating spherical or spheroidal cavity has received considerable attention for a very long time. The mathematical problem describing the inviscid inertial wave and oscillation was formulated more than a century ago by Poincaré in 1885. Bryan (1889) obtained a general implicit solution for the inviscid inertial modes using modified oblate spheroidal coordinates (see also Lyttleton 1953). Stewartson & Roberts (1963) considered the flow in an oblate cavity of a precessing rigid body and calculated the viscous correction for a particular inertial wave mode, the spin-over mode. Greenspan (1964) discussed many important properties of inertial waves in rotating fluids. Roberts & Stewartson (1965) examined an initial-value problem of a rotating spheroid in which the axis of rotation is impulsively moved. They presented a detailed analysis on the effect of viscosity through the Ekman boundary layer in an oblate spheroidal cavity. For an account of the earlier results and more references, see the excellent monograph by Greenspan (1968).

The stability properties of the Poincaré constant-vorticity solution in an oblate spheroid was examined in detail by Kerswell (1993). He also studied the effect of an azimuthal magnetic field on the Poincaré mode and estimated its Ohmic and viscous decay rates (Kerswell 1994; see also Malkus 1967). By accurately solving the full governing equations numerically, Hollerbach & Kerswell (1995) obtained the viscous decay rate of the Poincaré inertial mode (the spin-over mode) in spherical geometry, demonstrating convincingly that the breakdown of the oscillatory Ekman layer at critical latitudes, which leads to the weak internal shear layers (Rieutord & Valdettaro 1997), does not significantly affect the viscous decay rate. An explicit general analytical expression for all the inertial waves in a rotating sphere was found by Zhang *et al.* (2001) and the viscous effect on the spherical inertial waves was discussed by Liao, Zhang & Earnshaw (2001).

The problem of inertial waves and oscillations in a rapidly rotating spheroidal cavity was briefly discussed by Kudlick (1966, pp. 32–39). The main result of his investigation was summarized by Greenspan (1968). The focus of Kudlick's discussion was on a numerical procedure that may be used to compute an explicit solution for the Poincaré equation in spheroidal coordinates (see § 2.4 this paper for detail). However, a key stage of the procedure requires an analytical expression for the N real and distinct roots of a polynomial of degree N, where N is an important integer parameter in the solution. A small N usually corresponds to an inertial mode with the simple spatial structure. For example, the spin-over mode, which represents the spatially simplest inertial wave, is given by $N = 0$. When $N \leqslant 2$, an explicit solution in spheroidal coordinates for the inertial wave may be derived by using Kudlick's procedure. When $3 \leqslant N \leqslant 4$, the procedure cannot be practically used because the analytical expression for the N distinct roots becomes too complicated. When $N > 4$, Kudlick's procedure cannot be used because of the non-existence of an analytic expression for the N distinct roots. In consequence, Kudlick (1966) restricted his analysis to several modes with $1 \leqslant N \leqslant 2$ and the azimuthal wavenumbers $1 \leqslant m \leqslant 2$ in his investigation of spheroidal inertial waves.

We believe that this paper represents a major breakthrough in this classical problem. For the first time, we have found an explicit general analytical solution for a possibly complete set of the eigenfunctions ($0 \leqslant N < \infty$ and $0 \leqslant m < \infty$) representing non-axisymmetric inertial waves and axisymmetric inertial oscillations in an oblate spheroid of arbitrary eccentricity using explicit spheroidal coordinates in closed form. An explicit general solution for the spheroidal Ekman boundary layer near the surface of the cavity valid for all the inertial modes is derived to investigate the effect of viscosity. It is also for the first time that an explicit general analytical expression for the viscous corrections of all the inertial modes is found. On the basis of the explicit general solution, we have discovered and proved some unusual and intriguing properties of the problem which are shown in Appendix A.

We consider an incompressible viscous fluid contained in an oblate spheroidal cavity which rotates with constant angular velocity $\boldsymbol{e}_z\Omega$, where $\boldsymbol{e}_z$ is a unit vector. The envelope of the spheroidal cavity, S, is described by the equation

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1, \tag{1.1}$$

where a and b are the major and minor axes of an oblate spheroid ($a > b$), the z-axis is parallel to the axis of rotation. An important parameter describing the geometry of the oblate spheroid is its eccentricity defined as

$$\epsilon=\sqrt{\frac{(a^2-b^2)}{a^2}} \quad (0<\epsilon<1). \tag{1.2}$$

The limit $\epsilon \to 0$ corresponds to a special case for a sphere. The problem of inertial waves or oscillations in an incompressible viscous rotating fluid spheroid with constant kinematic viscosity ν and uniform density ρ is governed by the following dimensionless equations of the conservation of momentum and mass,

$$\frac{\partial \boldsymbol{u}}{\partial t}+R_o\boldsymbol{u}\cdot\nabla\boldsymbol{u}+2\boldsymbol{e}_z\times\boldsymbol{u}=-\nabla p+E\nabla^2\boldsymbol{u}+\boldsymbol{f}, \tag{1.3}$$

$$\nabla\cdot\boldsymbol{u}=0, \tag{1.4}$$

where $\boldsymbol{u}$ represents the velocity of three-dimensional flow, $\boldsymbol{f}$ is an external force and p is the reduced pressure in which the centrifugal acceleration is included. The Ekman number E, which provides the measure of relative importance between the typical viscous force and the Coriolis force, is defined as

$$E=\frac{\nu}{\Omega a^2}, \tag{1.5}$$

which is usually extremely small, $E \ll 1$, for a planetary fluid core (Gubbins & Roberts 1987). The Rossby number defined as

$$R_o=\frac{U}{\Omega a}, \tag{1.6}$$

where U is a typical amplitude of the flow, provides an estimate for the importance of the nonlinear term in (1.3). Furthermore, we have employed the following scales

$$\boldsymbol{r}\to a\boldsymbol{r}, \quad t\to t\Omega^{-1}, \boldsymbol{u}\to U\boldsymbol{u}, \quad p\to p\rho aU\Omega,$$

for length, time, velocity and pressure, respectively. In this study, the external force $\boldsymbol{f}$ in (1.3) is assumed to be zero, corresponding to an unforced inertial wave problem (Greenspan 1968; Hollerbach & Kerswell 1995). We also assume the non-slip boundary condition,

$$\boldsymbol{u} = 0, \tag{1.7}$$

on the envelope of the spheroidal cavity S. We further assume that the Rossby number R_o is sufficiently small so that the nonlinear term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ in (1.3) can be neglected.

Multiplying (1.3) with the velocity $\boldsymbol{u}$ and integrating the resulting equation over the spheroidal cavity V yield an energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_V |\boldsymbol{u}|^2\,\mathrm{d}V\right) = E\int_V (\boldsymbol{u}\cdot\nabla^2\boldsymbol{u})\,\mathrm{d}V < 0, \tag{1.8}$$

provided that the non-slip condition (1.7) for the velocity is satisfied. The three-dimensional integral on the right-hand side of (1.8), which is associated with the dissipation function and is usually referred to as the dissipation integral, is generally non-zero and negative. When $\boldsymbol{u}$ represents the velocity of a three-dimensional inviscid inertial wave in a rotating sphere, however, it was shown by Zhang *et al.* (2001) that

$$\int_V (\boldsymbol{u}\cdot\nabla^2\boldsymbol{u})\,\mathrm{d}V \equiv 0. \tag{1.9}$$

It should be pointed out that the terminology, the dissipation integral, adopted by Zhang *et al.* (2001) in the case of a sphere, may be confusing. Obviously, the unphysical result represented by (1.9) is a consequence of the unphysical tangential boundary condition for an inviscid inertial wave solution. Of course, the total viscous dissipation cannot vanish if a velocity $\boldsymbol{u}$ in a spherical or spheroidal cavity satisfies an appropriate physical tangential boundary condition such as the non-slip or the stress-free. In this sense, (1.9) represents an intriguing and unusual mathematical property of the fluid motion in the form of an inertial wave. This paper extends the result (1.9) obtained in a sphere to a spheroid cavity, showing that (1.9) also holds for all the inertial waves or oscillations in an oblate spheroid of arbitrary eccentricity.

In what follows, we shall first present a brief mathematical formulation in spheroidal polar coordinates in § 2. An explicit solution for inertial oscillation modes in spheroidal polar coordinates is derived in § 3 and an general explicit solution for inertial waves in spheroidal polar coordinates is derived in § 4. In § 5, we discuss a general spheroidal Ekman boundary solution near the surface of the spheroidal cavity. Section 6 examines the viscous effect on the inertial waves and concluding remarks are given in § 7.

2. Governing equations in spheroidal polar systems

2.1. *The coordinates transformation*

Before presenting the detailed analysis in spheroidal polar coordinates, it is desirable to discuss briefly the relevant metric tensor and the transformation between different coordinates used in our analysis. We employ three coordinate systems: cylindrical polar coordinates (s, ϕ, z), spheroidal polar coordinates (η, ϕ, τ) and modified spheroidal coordinates (X, ϕ, Y). The three coordinate systems are defined by the following relationships with the rectangular Cartesian coordinates (x, y, z)

$$x^2 = s^2\cos^2\phi = (\epsilon^2 + \eta^2)(1 - \tau^2)\cos^2\phi, \tag{2.1}$$

$$y^2 = s^2\sin^2\phi = (\epsilon^2 + \eta^2)(1 - \tau^2)\sin^2\phi, \tag{2.2}$$

$$z^2 = \eta^2\tau^2, \tag{2.3}$$

and

$$x^2 = A^2(1 - X^2)(1 - Y^2)\cos^2\phi, \tag{2.4}$$
$$y^2 = A^2(1 - X^2)(1 - Y^2)\sin^2\phi, \tag{2.5}$$
$$z^2 = B^2X^2Y^2, \tag{2.6}$$

where A and B are a function of both the half-frequency, σ, of an inertial wave and the eccentricity of a spheroid,

$$A^2 = \frac{1 - \epsilon^2\sigma^2}{1 - \sigma^2}, \quad B^2 = \frac{1 - \epsilon^2\sigma^2}{\sigma^2} \quad (-1 < \sigma < 1). \tag{2.7}$$

The spheroidal envelope S of the cavity in spheroidal polar coordinates is described by $\eta = \sqrt{(1 - \epsilon^2)}$. Modified spheroidal polar coordinates (X, ϕ, Y) are used in the process of deriving an implicit solution while spheroidal polar coordinates (η, ϕ, τ) are adopted in the analysis of spheroidal Ekman boundary layers and in the evaluation of the viscous correction of an inertial wave or oscillation. The analysis frequently requires the transformation between different coordinates. Let us define a vector $\boldsymbol{Q}$ as follows

$$\boldsymbol{Q} = Q_x\boldsymbol{e}_x + Q_y\boldsymbol{e}_y + Q_z\boldsymbol{e}_z = Q_s\boldsymbol{e}_s + Q_\phi\boldsymbol{e}_\phi + Q_z\boldsymbol{e}_z = Q_\eta\boldsymbol{e}_\eta + Q_\phi\boldsymbol{e}_\phi + Q_\tau\boldsymbol{e}_\tau, \tag{2.8}$$

where $\boldsymbol{e}_x, \ldots, \boldsymbol{e}_\tau$ denote unit vectors in the corresponding coordinates. A central element in deriving the coordinates transformation is the fundamental metric tensor g_{ij} for spheroidal polar coordinates

$$\left.\begin{aligned} g_{11} = g_{\eta\eta} &= \frac{\eta^2 + \epsilon^2\tau^2}{\eta^2 + \epsilon^2}, \\ g_{22} = g_{\phi\phi} &= (\eta^2 + \epsilon^2)(1 - \tau^2), \\ g_{33} = g_{\tau\tau} &= \frac{\eta^2 + \epsilon^2\tau^2}{1 - \tau^2}, \\ g_{ij} &= 0 \quad \text{if} \quad i \neq j. \end{aligned}\right\} \tag{2.9}$$

The determinant of the metric tensor is

$$g = \det[g_{ij}] = (\eta^2 + \epsilon^2\tau^2)^2. \tag{2.10}$$

Making use of the fundamental metric tensor g_{ij}, we can establish that

$$[Q_\eta, Q_\phi, Q_\tau]^T = \boldsymbol{D}[Q_x, Q_y, Q_z]^T = \boldsymbol{DM}[Q_s, Q_\phi, Q_z]^T, \tag{2.11}$$

where matrices $\boldsymbol{D}$ and $\boldsymbol{M}$ of sizes 3×3 are

$$\boldsymbol{D} = \frac{1}{\sqrt{\eta^2 + \epsilon^2\tau^2}} \begin{bmatrix} \eta\cos\phi\sqrt{1 - \tau^2} & \eta\sin\phi\sqrt{1 - \tau^2} & \tau\sqrt{\eta^2 + \epsilon^2} \\ -\sin\phi\sqrt{\eta^2 + \epsilon^2\tau^2} & \cos\phi\sqrt{\eta^2 + \epsilon^2\tau^2} & 0 \\ -\tau\cos\phi\sqrt{\eta^2 + \epsilon^2} & -\tau\sin\phi\sqrt{\eta^2 + \epsilon^2} & \eta\sqrt{1 - \tau^2} \end{bmatrix}$$

and

$$\boldsymbol{M} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Based on the metric tensor g_{ij} and (2.11), for example, we can transform the unit vector $\boldsymbol{e}_z$ in cylindrical polar coordinates into a unit vector in spheroidal polar coordinates

$$\boldsymbol{e}_z = (\hat{z}_1, \hat{z}_2, \hat{z}_3) = (\hat{z}_\eta, \hat{z}_\phi, \hat{z}_\tau) = \left(\frac{\tau\sqrt{\epsilon^2+\eta^2}}{\sqrt{\eta^2+\epsilon^2\tau^2}}, 0, \frac{\eta\sqrt{1-\tau^2}}{\sqrt{\eta^2+\epsilon^2\tau^2}} \right); \tag{2.12}$$

we can express the terms such as $\boldsymbol{e}_z \times \boldsymbol{u}$ and $\nabla \times \boldsymbol{u}$ using spheroidal polar coordinates

$$(\boldsymbol{e}_z \times \boldsymbol{u})_k = \sum_{l=1}^{3}\sum_{m=1}^{3} \frac{\sqrt{g_{ll}g_{mm}g_{kk}}}{\sqrt{g}} e_{klm}\hat{z}_l u_m \quad (k = 1, 2, 3), \tag{2.13}$$

$$(\nabla \times \boldsymbol{u})_k = \left[\sum_{l=1}^{3}\sum_{m=1}^{3} e_{klm} \frac{\partial}{\partial x_l}(u_m \sqrt{g_{mm}}) \right] \frac{\sqrt{g_{kk}}}{\sqrt{g}} \quad (k = 1, 2, 3), \tag{2.14}$$

where e_{klm} is the permutation symbol of rank three and

$$x_1 = \eta, u_1 = \boldsymbol{e}_\eta \cdot \boldsymbol{u} = u_\eta; \quad x_2 = \phi, u_2 = \boldsymbol{e}_\phi \cdot \boldsymbol{u} = u_\phi; \quad x_3 = \tau, u_3 = \boldsymbol{e}_\tau \cdot \boldsymbol{u} = u_\tau.$$

Although the three coordinate systems are employed in various stages of the analysis, we shall present our results mainly using spheroidal polar coordinates which are the natural coordinates for a spheroidal problem and have to be adopted when the effect of viscosity is studied.

As displayed in the metric tensor g_{ij} for spheroidal polar coordinates, there are three terms, $\sqrt{\eta^2+\epsilon^2}$, $\sqrt{1-\tau^2}$ and $\sqrt{\eta^2+\epsilon^2\tau^2}$, which appear frequently in our analytical solutions. To simplify the presentation, we introduce the following notation

$$u = \sqrt{\eta^2+\epsilon^2}, \quad v = \sqrt{1-\tau^2}, \quad w = \sqrt{\eta^2+\epsilon^2\tau^2}. \tag{2.15}$$

2.2. *Governing equations and perturbations*

Using the fundamental metric tensor g_{ij} and neglecting the nonlinear term in (1.3) controlled by the Rossby number, we can write the governing equations (1.3)–(1.4) in spheroidal polar coordinates

$$w\frac{\partial u_\eta}{\partial t} - 2\eta v u_\phi + u\frac{\partial p}{\partial \eta} = \frac{E}{u}\left[\frac{\partial}{\partial \tau}(uv\Omega_\phi) - \frac{\partial}{\partial \phi}\left(\frac{w\Omega_\tau}{v}\right) \right], \tag{2.16}$$

$$w\frac{\partial u_\tau}{\partial t} + 2\tau u u_\phi + v\frac{\partial p}{\partial \tau} = \frac{E}{v}\left[\frac{\partial}{\partial \phi}\left(\frac{w\Omega_\eta}{u}\right) - \frac{\partial}{\partial \eta}(uv\Omega_\phi) \right], \tag{2.17}$$

$$w\frac{\partial u_\phi}{\partial t} + 2(\eta v u_\eta - \tau u u_\tau) + \frac{w}{uv}\frac{\partial p}{\partial \phi} = \frac{Euv}{w}\left[\frac{\partial}{\partial \eta}\left(\frac{w\Omega_\tau}{v}\right) - \frac{\partial}{\partial \tau}\left(\frac{w\Omega_\eta}{u}\right) \right], \tag{2.18}$$

$$uv\left[\frac{\partial}{\partial \eta}(wuu_\eta) + \frac{\partial}{\partial \tau}(wvu_\tau) \right] + w^2\frac{\partial u_\phi}{\partial \phi} = 0, \tag{2.19}$$

where

$$\begin{aligned} \Omega_\eta &= \frac{1}{uv}\frac{\partial u_\tau}{\partial \phi} - \frac{1}{w}\frac{\partial (vu_\phi)}{\partial \tau}, \\ \Omega_\tau &= \frac{1}{w}\frac{\partial (uu_\phi)}{\partial \eta} - \frac{1}{uv}\frac{\partial u_\eta}{\partial \phi}, \\ \Omega_\phi &= \frac{v}{w^2}\frac{\partial (wu_\eta)}{\partial \tau} - \frac{u}{w^2}\frac{\partial (wu_\tau)}{\partial \eta}. \end{aligned} \tag{2.20}$$

The assumption of the non-slip boundary condition (1.7) on the surface of the spheroidal container S corresponds to

$$u_\eta = u_\phi = u_\eta = 0 \quad \text{at} \quad \eta = \sqrt{1-\epsilon^2}. \tag{2.21}$$

Clearly, in the limiting case, $\epsilon \to 0$, for a sphere, η in spheroidal polar coordinates becomes identical to r in spherical polar coordinates. This paper is mainly concerned with solutions of (2.16)–(2.19) subject to the non-slip boundary condition (2.21) in a rapidly rotating spheroid of arbitrary eccentricity.

It is important to note that the amplitude of the Ekman boundary-layer flow with a non-slip boundary condition is of the same order as that of the mainstream flow. In consequence, an explicit oscillatory boundary-layer solution is required in order to determine the matching condition between the boundary layer and interior solutions. The general idea for the asymptotic theory of inertial waves was discussed by Greenspan (1968). For a sufficiently small E, fluid motions within a rotating spheroidal cavity can be separated into the internal flow $\boldsymbol{u}_i$ and the boundary-layer flow $\boldsymbol{u}_b$. We may expand both the internal and boundary-layer flows in terms of small but non-zero E

$$\boldsymbol{u}_i = \left[\boldsymbol{u}_0(\eta,\tau) + E^{1/2}\boldsymbol{u}_1(\eta,\tau) + \cdots\right] \mathrm{e}^{\mathrm{i}(\omega t + m\phi)}, \tag{2.22}$$

$$p_i = \left[p_0(\eta,\tau) + E^{1/2}p_1(\eta,\tau) + \cdots\right] \mathrm{e}^{\mathrm{i}(\omega t + m\phi)}, \tag{2.23}$$

$$\boldsymbol{u}_b = \left[\hat{\boldsymbol{u}}_0(\eta,\tau) + E^{1/2}\hat{\boldsymbol{u}}_1(\eta,\tau) + \cdots\right] \mathrm{e}^{\mathrm{i}(\omega t + m\phi)}, \tag{2.24}$$

$$\omega = 2\sigma - \mathrm{i}GE^{1/2} + \cdots, \tag{2.25}$$

where $m \geqslant 0$ is the azimuthal wavenumber of an inertial mode, $\mathrm{i} = \sqrt{-1}$, σ is the half-frequency of a non-dissipative inertial wave and G represents the correction of the frequency of a non-dissipative inertial wave due to the effect of viscosity.

2.3. *The Poincaré equation in different coordinate systems*

The leading-order mainstream problem describes non-dissipative inertial waves in a rotating spheroidal cavity. Its governing equations are obtained by substituting the expansions (2.22)–(2.23) and (2.25) into (2.16)–(2.19) and taking the leading-order terms,

$$2\mathrm{i}\sigma w u_{\eta 0} - 2\eta v u_{\phi 0} + u\frac{\partial p_0}{\partial \eta} = 0, \tag{2.26}$$

$$2\mathrm{i}\sigma w u_{\tau 0} + 2\tau u u_{\phi 0} + v\frac{\partial p_0}{\partial \tau} = 0, \tag{2.27}$$

$$2\mathrm{i}\sigma w u v u_{\phi 0} + 2(\eta u v^2 u_{\eta 0} - \tau u^2 v u_{\tau 0}) + \mathrm{i}m w p_0 = 0, \tag{2.28}$$

$$uv\left[\frac{\partial}{\partial \eta}\left(w u u_{\eta 0}\right) + \frac{\partial}{\partial \tau}\left(w v u_{\tau 0}\right)\right] + \mathrm{i}m w^2 u_{\phi 0} = 0, \tag{2.29}$$

where

$$\boldsymbol{u}_0(\eta,\tau) = u_{\eta 0}\boldsymbol{e}_\eta + u_{\phi 0}\boldsymbol{e}_\phi + u_{\tau 0}\boldsymbol{e}_\tau.$$

The only boundary condition required in the leading approximation is

$$u_{\eta 0} = 0 \quad \text{at} \quad \eta = \sqrt{1-\epsilon^2}. \tag{2.30}$$

For a further simplification in the presentation we introduce

$$(u_{\eta 0}, u_{\phi 0}, u_{\tau 0})(\eta,\tau) = (\mathrm{i}V_\eta, V_\phi, \mathrm{i}V_\tau)(\eta,\tau) \tag{2.31}$$

so that V_η, V_ϕ and V_τ are real functions of η and τ.

It is well known that the elimination of the velocity components from (2.26)–(2.29) leads to the Poincaré equation for the pressure p_0, which may be written in three different forms. In cylindrical polar coordinates (s, ϕ, z), the Poincaré equation has the form (Greenspan 1968)

$$\left(\frac{1}{s}\frac{\partial p_0}{\partial s} + \frac{\partial^2 p_0}{\partial s^2} - \frac{m^2}{s^2} p_0\right) + \frac{(\sigma^2 - 1)}{\sigma^2}\frac{\partial^2 p_0}{\partial z^2} = 0. \tag{2.32}$$

In spheroidal polar coordinates (η, ϕ, τ), the Poincaré equation has the form

$$C_{\tau\tau}\frac{\partial^2 p_0}{\partial \tau^2} + C_{\eta\eta}\frac{\partial^2 p_0}{\partial \eta^2} + C_\tau \frac{\partial p_0}{\partial \tau} + C_\eta \frac{\partial p_0}{\partial \eta} + C_{\tau\eta}\frac{\partial^2 p_0}{\partial \tau \partial \eta} + C_0 p_0 = 0, \tag{2.33}$$

where

$$\begin{aligned}
C_{\tau\tau} &= \frac{v^2}{w^4}\left(\tau^2 u^2 + \frac{(\sigma^2 - 1)}{\sigma^2}\eta^2 v^2\right), \\
C_{\eta\eta} &= \frac{u^2}{w^4}\left(\eta^2 v^2 + \frac{(\sigma^2 - 1)}{\sigma^2}\tau^2 u^2\right), \\
C_\tau &= \frac{\tau}{w^6}[-2w^4 + u^2 v^2(3\eta^2 - \tau^2\epsilon^2)] \\
&\quad + \frac{(\sigma^2 - 1)}{\sigma^2}\left(\frac{v^2}{\tau w^6}\right)[w^4 - \eta^2 w^2(1 + \tau^2) - 2\eta^2\tau^2(u^2 + \epsilon^2 v^2)], \\
C_\eta &= \frac{\eta}{w^6}[2w^4 + u^2 v^2(3\tau^2\epsilon^2 - \eta^2)] \\
&\quad + \frac{(\sigma^2 - 1)}{\sigma^2}\left(\frac{u^2}{\eta w^6}\right)[w^4 - \tau^2 w^2(\epsilon^2 - \eta^2) - 2\eta^2\tau^2(u^2 + \epsilon^2 v^2)], \\
C_{\tau\eta} &= -\frac{2\tau\eta u^2 v^2}{\sigma^2 w^4}, \\
C_0 &= -\frac{m^2}{u^2 v^2}.
\end{aligned}$$

In modified spheroidal polar coordinates (X, Y), Bryan (1889) showed that the Poincaré equation can be written as

$$\frac{\partial}{\partial X}\left[(1 - X^2)\frac{\partial p_0}{\partial X}\right] - \frac{\partial}{\partial Y}\left[(1 - Y^2)\frac{\partial p_0}{\partial Y}\right] = \left[\frac{m^2}{(1 - X^2)} - \frac{m^2}{(1 - Y^2)}\right] p_0. \tag{2.34}$$

Bryan (1889) recognized the separable solution for (2.34) in the form

$$p_0 = P_l^m(X(\eta, \tau))P_l^m(Y(\eta, \tau)) \quad (l \geqslant m), \tag{2.35}$$

where P_l^m is an associated Legendre function of the first kind and

$$X(\eta, \tau) = \frac{1}{\sqrt{2}AB}[\Delta + \sqrt{\Delta^2 - (2A^2 B\eta\tau)^2}]^{1/2}, \tag{2.36}$$

$$Y(\eta, \tau) = \frac{1}{\sqrt{2}AB}[\Delta - \sqrt{\Delta^2 - (2A^2 B\eta\tau)^2}]^{1/2}, \tag{2.37}$$

$$\Delta = (AB)^2 + (A\eta\tau)^2 - (Buv)^2. \tag{2.38}$$

It is of importance to note that, in order to find solutions of (2.16)–(2.19) subject to the non-slip boundary condition (2.21), we require solutions of the Poincaré equation (2.33) in terms of explicit spheroidal coordinates (η, τ).

2.4. *Kudlick's procedure*

The Poincaré equation in the form (2.33) in spheroidal coordinates (η, τ) is too complicated to solve directly. However, the implicit solution (2.35) using modified spheroidal polar coordinates (X, Y), though mathematically compact, is not practically useful because of the complex relationship (2.36)–(2.38). Upon realizing the limitation of Bryan's implicit solution (2.35), Kudlick (1966) proposed a procedure to compute an explicit solution of the Poincaré equation in cylindrical coordinates. For equatorially symmetric inertial waves (even modes), for example, he showed that p_0 can be expressed in the form of a summation of a double polynomial

$$p_0(s, z) = \frac{s^m}{\alpha^*} \prod_{k=1}^{N} (D_k + A_k s^2 + B_k z^2), \tag{2.39}$$

where

$$\begin{aligned}
\alpha^* &= \frac{1+\epsilon^*(1-\sigma^2)}{(1+\epsilon^*)(1-\sigma^2)}, \\
\epsilon^* &= \frac{\epsilon}{\sqrt{1-\epsilon^2}}, \\
D_k &= x_k(x_k - 1), \\
A_k &= \left(\frac{1+\epsilon^*}{1+\epsilon^*(1-\sigma^2)} \right) x_k(1-\sigma^2), \\
B_k &= \left(\frac{1+\epsilon^*}{1+\epsilon^*(1-\sigma^2)} \right) \sigma^2(1-x_k),
\end{aligned}$$

and $x_k, k = 1, 2, \ldots, N$, are the N distinct and real roots of the equation

$$\sum_{j=0}^{N} (-1)^j \frac{[2(2N+m-j)]!}{j![2(N-j)]!(2N+m-j)!} x_k^{N-j} = 0 \tag{2.40}$$

exclusive of zero and one.

Kudlick's procedure consists of the three steps: (i) solve (2.40) to find the N real and distinct roots, $x_k, k = 1, 2, \ldots, N$, (ii) substitute the N roots into (2.39) to obtain p_0, and (iii) calculate solutions such as the viscous decay factors using p_0 which is a function of the roots $x_k, k = 1, 2, \ldots, N$.

The crucial step in the procedure involves solving (2.40) to obtain an analytical expression for the N real roots, $x_k, k = 1, 2, \ldots, N$. When $3 \leqslant N \leqslant 4$, an analytical expression for the N roots is already too lengthy to be practically useful. Kudlick (1966) therefore restricted his analysis to $N \leqslant 2$. When $N > 4$, the procedure cannot be used because the analytical expression for the roots does not exist. In general, coefficients for the double polynomial, $(D_k, A_k, B_k, k = 1, 2, \ldots, N)$, must be computed numerically to find a numerical solution of the Poincaré equation. In short, Kudlick's procedure (1966) (see also §2.12, Greenspan 1968) cannot be used to find the general explicit analytical solution $(0 \leqslant N < \infty)$.

2.5. *Symmetries of a spheroidal inertial mode*

Solutions of (2.26)–(2.29) in a rotating spheroid have two different spatial symmetries with respect to a meridional plane. An axisymmetric solution is characterized by the symmetry property

$$(u_\eta, u_\phi, u_\tau, p)(\eta, \tau, \phi) = (u_\eta, u_\phi, u_\tau, p)(\eta, \tau, \phi + \phi_a), \tag{2.41}$$

where ϕ_a is an arbitrary constant, while a non-axisymmetric solution satisfies

$$(u_\eta, u_\phi, u_\tau, p)(\eta, \tau, \phi) = (u_\eta, u_\phi, u_\tau, p)(\eta, \tau, \phi + 2\pi/m). \tag{2.42}$$

In general, an axisymmetric solution in a rotating spheroid represents oscillatory fluid motions which are usually referred to as inertial oscillations while a non-axisymmetric solution represents azimuthally travelling waves which are referred to as inertial waves (Greenspan 1968). There are important physical and mathematical differences between the oscillation ($m = 0$) and wave ($m \geqslant 1$) solutions. Generally speaking, we cannot obtain an oscillation mode ($m = 0$) from a wave mode ($m \geqslant 1$) simply by letting $m = 0$. For a clearer discussion, we shall present the solutions of axisymmetric oscillations and inertial waves separately.

There also exist two different parities of the solutions with respect to the equatorial plane, $\tau = 0$, of a rotating spheroid. An equatorially symmetric solution is characterized by the symmetry property

$$(u_\eta, u_\phi, u_\tau, p)(\eta, \tau, \phi) = (u_\eta, u_\phi, -u_\tau, p)(\eta, -\tau, \phi), \tag{2.43}$$

while an equatorially antisymmetric wave satisfies

$$(u_\eta, u_\phi, u_\tau, p)(\eta, \tau, \phi) = (-u_\eta, -u_\phi, u_\tau, -p)(\eta, -\tau, \phi). \tag{2.44}$$

The mathematical analyses for both the equatorial parities are nearly identical except for shifting several integer indices. In principle, it is feasible to construct a single mathematical expression valid for both the parities by introducing some additional parameters. However, the mathematical expression for our general explicit solution is already complicated, involving many different indices. For a clearer representation, we shall write the solutions with different equatorial symmetries separately.

3. Explicit solutions for axisymmetric oscillations

We shall take three steps to derive a general explicit solution for the spheroidal axisymmetric inertial oscillations. First, modified spheroidal coordinates (2.36)–(2.37) are used to obtain an implicit oscillation solution of (2.34), denoted by $\bar{p}_0$, as a function of X and Y. In the second step, several explicit solutions for $\bar{p}_0$ as a function of η and τ are derived, which offer an essential clue and pattern leading to the general explicit solution for $\bar{p}_0$. Finally, we derive the general explicit solution of the three velocity components for all the axisymmetric oscillation modes by solving (2.26)–(2.29) for a given $\bar{p}_0$.

Equations (2.34)–(2.35) suggest that the equatorially symmetric implicit solution for $\bar{p}_0$ can be written as

$$\bar{p}_0 = \sum_{i=0}^{N}\sum_{j=0}^{N}\left[\frac{(-1)^{i+j}}{i!j!}\right][X^{2N-2j}Y^{2N-2i}]\left[\frac{[2(2N-i)]![2(2N-j)]!}{(2N-i)!(2N-j)![2(N-i)]![2(N-j)]!}\right], \tag{3.1}$$

where $N \geqslant 2$, $n! = n(n-1)(n-2)\cdots 1$ and $0! = 1$. An arbitrary normalization is used in (3.1). Similarly, the equatorially antisymmetric implicit solution for $\bar{p}_0$ is

$$\bar{p}_0 = (\tau\eta)\sum_{i=0}^{N}\sum_{j=0}^{N}\left[\frac{(-1)^{i+j}}{i!j!}\right][X^{2N-2j}Y^{2N-2i}] \times \left[\frac{[2(2N-i+1)]![2(2N-j+1)]!}{(2N-i+1)!(2N-j+1)![2(N-i)+1]![2(N-j)+1]!}\right], \tag{3.2}$$

where $N \geqslant 1$. The implicit solution for $\bar{p}_0$ given by (3.1)–(3.2) is not practically useful, both mathematically and computationally, because of the complicated relationship given by (2.36)–(2.37). In order to study the effect of viscosity on inertial oscillations, it is necessary to find the solution that is expressed explicitly in terms of spheroidal coordinates η and τ. In other words, we require a general explicit analytical solution for the Poincaré equation (2.33).

The explicit solution for axisymmetric inertial oscillations can be divided into the following ten classes according to its equatorial symmetries and spatial complexities:

(i) Class $\bar{\mathscr{A}}_1$: symmetries (2.41) and (2.44) with $N = 1$;
(ii) Class $\bar{\mathscr{S}}_2$: symmetries (2.41) and (2.43) with $N = 2$;
(iii) Class $\bar{\mathscr{A}}_2$: symmetries (2.41) and (2.44) with $N = 2$;
(iv) Class $\bar{\mathscr{S}}_3$: symmetries (2.41) and (2.43) with $N = 3$;
(v) Class $\bar{\mathscr{A}}_3$: symmetries (2.41) and (2.44) with $N = 3$;
(vi) Class $\bar{\mathscr{S}}_4$: symmetries (2.41) and (2.43) with $N = 4$;
(vii) Class $\bar{\mathscr{A}}_4$: symmetries (2.41) and (2.44) with $N = 4$;
(viii) Class $\bar{\mathscr{S}}_5$: symmetries (2.41) and (2.43) with $N = 5$;
(ix) Class $\bar{\mathscr{A}}_G$: a general solution with symmetries (2.41) and (2.44), $N \geqslant 5$;
(x) Class $\bar{\mathscr{S}}_G$: a general solution with symmetries (2.41) and (2.43), $N \geqslant 6$.

For classes (i)–(viii), analytical solutions in closed form can be obtained; for classes (ix)–(x), exact values for the frequencies of axisymmetric oscillation modes cannot be found.

3.1. *Class* $\bar{\mathscr{A}}_1$

The simplest mode for equatorially antisymmetric inertial oscillation, class $\mathscr{A}_1$, offers an ideal example to illustrate the solution procedure of our analysis. A particular advantage is that the solution is sufficiently simple to allow us to write out all the relevant mathematical expressions in detail. The pressure $\bar{p}_0$ for class $\bar{\mathscr{A}}_1$ can be obtained by letting $N = 1$ in (3.2) and using the transformation laws (2.36)–(2.37), which yields

$$\bar{p}_0 = \tfrac{3}{2}\eta\tau - 3u^2v^2\eta\tau - \frac{1}{2(1-\epsilon^2)}(\eta\tau)^3, \tag{3.3}$$

where $\bar{p}_0$ is normalized such that it is consistent with the general solution for $N > 5$. With the availability of an explicit expression for p_0 in terms of η and τ, we then derive the three velocity components of the axisymmetric oscillation in spheroidal polar coordinates by using (2.26)–(2.29)

$$V_\eta = \frac{3u}{4w}\left[\frac{5\sigma v^2\tau\eta^2}{1-\epsilon^2\sigma^2} + \frac{\tau}{\sigma} - \frac{5(1-\sigma^2)u^2v^2\tau}{2\sigma(1-\epsilon^2\sigma^2)} - \frac{5\sigma\tau^3\eta^2}{1-\epsilon^2\sigma^2}\right], \tag{3.4}$$

$$V_\phi = -\frac{15}{4(1-\epsilon^2\sigma^2)}(uv\eta\tau), \tag{3.5}$$

ϵ	2σ	Re[G]	Im[G]
0.1	0.898027	−3.2550	0.2916
$\sqrt{3}/2$	1.414214 (1.414)	−4.2051 (−4.205)	1.5838 (0.1584)
0.9	1.507557	−4.3625	1.9044

TABLE 1. Class $\bar{\mathscr{A}}_1$ examples.

$$V_\tau = \frac{3v}{4w}\left[-\frac{5\sigma u^2\tau^2\eta}{1-\epsilon^2\sigma^2} + \frac{\eta}{\sigma} - \frac{5(1-\sigma^2)u^2v^2\eta}{2\sigma(1-\epsilon^2\sigma^2)} - \frac{5\sigma\tau^2\eta^3}{1-\epsilon^2\sigma^2}\right]. \tag{3.6}$$

To determine the half-frequency σ of the oscillation, we demand the vanishing of the normal velocity V_η, given by (3.4), at the envelope of the spheroidal cavity $\eta = \sqrt{1-\epsilon^2}$. This results in an equation for σ as a function of ϵ

$$\sigma = \pm\left(\frac{1}{5-4\epsilon^2}\right)^{1/2}. \tag{3.7}$$

It follows that, for a given ϵ, there exist two axisymmetric oscillation modes in this class. Several examples for the positive half-frequency σ, together with the corresponding viscous corrections (see § 6 for detail; the numbers in parentheses were given by Kudlick 1966) are shown in table 1. It is worth noting that both $\pm|\sigma|$ are the solution of axisymmetric oscillations and they are related by

$$(V_\eta, V_\phi, V_\tau, \bar{p}_0)(\sigma) = (-V_\eta, V_\phi, -V_\tau, \bar{p}_0)(-\sigma). \tag{3.8}$$

As a result, we shall only focus on the solutions with $\sigma > 0$. Inserting (3.7) into (3.4)–(3.6), we obtain the three velocity components of the oscillation for this class in closed form

$$V_\eta = \frac{3u\sqrt{5-4\epsilon^2}}{4w}\left[\frac{v^2\tau\eta^2}{1-\epsilon^2} + \tau - 2u^2v^2\tau - \frac{\tau^3\eta^2}{1-\epsilon^2}\right], \tag{3.9}$$

$$V_\phi = -\frac{3(5-4\epsilon^2)}{4(1-\epsilon^2)}(uv\eta\tau), \tag{3.10}$$

$$V_\tau = \frac{3v\sqrt{5-4\epsilon^2}}{4w}\left[-\frac{u^2\tau^2\eta}{1-\epsilon^2} + \eta - 2u^2v^2\eta - \frac{\tau^2\eta^3}{1-\epsilon^2}\right]. \tag{3.11}$$

It should be pointed out that we have been unable to find such an explicit solution, though it represents the simplest one in our analysis, in the existing literature.

3.2. *Class* $\bar{\mathscr{S}}_2$

The simplest equatorially symmetric solution for axisymmetric oscillation can be obtained by letting $N = 2$ in (3.1) together with the transformation laws (2.36)–(2.37), which leads to

$$\bar{p}_0 = \tfrac{3}{4} - \left[\frac{15(1-\sigma^2)}{4(1-\epsilon^2\sigma^2)}\right](uv)^2 + \left[\frac{105(1-\sigma^2)^2}{32(1-\epsilon^2\sigma^2)^2}\right](uv)^4 - \left[\frac{15\sigma^2}{2(1-\epsilon^2\sigma^2)}\right](\eta\tau)^2 + \left[\frac{105\sigma^2(1-\sigma^2)}{4(1-\epsilon^2\sigma^2)^2}\right](uv\eta\tau)^2 + \left[\frac{35\sigma^4}{4(1-\epsilon^2\sigma^2)^2}\right](\eta\tau)^4. \tag{3.12}$$

ϵ	2σ	Re[G]	Im[G]
0.1	1.313064	−3.3903	0.4336
$\sqrt{3}/2$	1.732051 (1.732)	−3.8448 (−3.844)	1.3845 (1.384)
0.9	1.786474	−3.8975	1.5352

TABLE 2. Class $\bar{\mathscr{S}}_2$ examples.

By the same procedure as described for class $\bar{\mathscr{A}}_1$, we can derive the three velocity components of the oscillation, which is too lengthy to be spelled out and is contained in the general solution (3.24)–(3.26) by setting $N = 2$. The normal flow condition $V_\eta = 0$ at $\eta = \sqrt{1-\epsilon^2}$ gives

$$\sigma = \pm\left(\frac{3}{7-4\epsilon^2}\right)^{1/2}. \tag{3.13}$$

There exist two axisymmetric oscillation modes for a given ϵ. Several examples for 2σ together with the corresponding viscous corrections (see § 6) for class $\bar{\mathscr{S}}_2$ are given in table 2 for three different eccentricities. On substitution of (3.13) into expressions for the velocity (V_η, V_ϕ, V_τ) given by (3.24)–(3.26) at $N = 2$, and for the pressure p_0 given by (3.12), we obtain the fully explicit oscillation solution in closed form for class $\bar{\mathscr{S}}_2$.

3.3. *Class* $\bar{\mathscr{A}}_2$

When $N = 2$ with equatorial antisymmetry (2.44), class $\bar{\mathscr{A}}_2$, the axisymmetric oscillation solution in closed form can still be derived. Equation (3.2) at $N = 2$ with the transformation laws (2.36)–(2.37) gives

$$\begin{aligned}\bar{p}_0 = \tfrac{3}{4}(\eta\tau) - \frac{105(1-\sigma^2)}{4(1-\epsilon^2\sigma^2)}(u^2v^2\eta\tau) + \frac{945(1-\sigma^2)^2}{32(1-\epsilon^2\sigma^2)^2}(u^4v^4\eta\tau)\\ - \frac{35\sigma^2}{2(1-\epsilon^2\sigma^2)}(\eta^3\tau^3) + \frac{315\sigma^2(1-\sigma^2)}{4(1-\epsilon^2\sigma^2)^2}(u^2v^2\eta^3\tau^3) + \frac{63\sigma^4}{4(1-\epsilon^2\sigma^2)^2}(\eta^5\tau^5).\end{aligned} \tag{3.14}$$

From (2.26)–(2.29), we then derive the three velocity components (V_η, V_ϕ, V_τ) in spheroidal polar coordinates, which are again too lengthy to be spelled out here and can be obtained by letting $N = 2$ in (3.29)–(3.31). The normal flow condition that $V_\eta = 0$ at $\eta = \sqrt{1-\epsilon^2}$ yields

$$\sigma = \pm\left[\frac{(7-6\epsilon^2) \pm 2\sqrt{7}(1-\epsilon^2)}{(21-28\epsilon^2+8\epsilon^4)}\right]^{1/2}. \tag{3.15}$$

Several examples of the frequency 2σ and the corresponding viscous corrections for three different eccentricities are given in table 3.

It follows that there exist four axisymmetric oscillation modes for a given ϵ in this class. On substitution of (3.15) into the corresponding expressions for the pressure $\bar{p}_0$ given by (3.14) and the flow velocity given by (3.29)–(3.31) at $N = 2$, we obtain the fully explicit inertial wave solutions in closed form for this class.

3.4. *Class* $\bar{\mathscr{S}}_3$

When $N = 3$ with equatorial symmetry (2.44), class $\bar{\mathscr{S}}_3$, oscillation solutions in closed form can be also derived. By the same method, we can readily obtain an expression for $\bar{p}_0$ in spheroidal polar coordinates, and then derive the three velocity components

ϵ	2σ	Re[G]	Im[G]
0.1	0.573101	−4.4447	0.1956
0.1	1.533293	−3.4511	0.5123
$\sqrt{3}/2$	1.022904	−6.2601	1.5075
$\sqrt{3}/2$	1.843397 (1.843)	−3.6989 (−3.698)	1.1951 (1.194)
0.9	1.127699	−6.6513	1.9642
0.9	1.877612	−3.7234	1.2812

TABLE 3. Class $\bar{\mathscr{A}}_2$ examples.

ϵ	2σ	Re[G]	Im[G]
0.1	0.941377	−4.6518	0.3255
0.1	1.663034	−3.4828	0.5606
$\sqrt{3}/2$	1.455827	−5.8825	1.6812
$\sqrt{3}/2$	1.896010 (1.896)	−3.6302 (−3.629)	1.0720 (1.070)
0.9	1.545636	−6.0775	2.0032
0.9	1.919484	−3.6436	1.1285

TABLE 4. Class $\bar{\mathscr{S}}_3$ examples.

(V_η, V_ϕ, V_τ) from (2.26)–(2.29), which are again too lengthy to be spelled out here. They are contained in the general expressions (3.24)–(3.26) by letting $N = 3$. The normal flow condition that $V_\eta = 0$ at $\eta = \sqrt{1-\epsilon^2}$ yields

$$\sigma = \pm\left[\frac{5(3-2\epsilon^2) \pm \sqrt{60}(1-\epsilon^2)}{(33-36\epsilon^2+8\epsilon^4)}\right]^{1/2}. \tag{3.16}$$

Several examples of the frequency 2σ and the corresponding viscous corrections for class $\bar{\mathscr{S}}_3$ are shown in table 4 for three different eccentricities.

It follows that there exist four different oscillation modes for a given ϵ in this class. On substitution of (3.16) into the corresponding expressions for the pressure $\bar{p}_0$ given by (3.23) at $N = 3$ and the flow velocity given by (3.24)–(3.26) at $N = 3$, we obtain the fully explicit inertial oscillation solution in closed form for this class.

3.5. *Class $\bar{\mathscr{A}}_3$*

When $N = 3$ with equatorial antisymmetry (2.43), class $\bar{\mathscr{A}}_3$, the explicit solution becomes more complicated. However, the solution in closed form can still be derived. By the same method, we can obtain an explicit expression for $\bar{p}_0(\eta,\tau)$ and then derive the three velocity components (V_η, V_ϕ, V_τ) in spheroidal coordinates, which are contained in the general expressions (3.28)–(3.31) by letting $N = 3$. The normal flow condition that $V_\eta = 0$ at $\eta = \sqrt{1-\epsilon^2}$ yields an equation for σ

$$\Lambda_1\sigma^6 + (240\epsilon^4 - 720\epsilon^2 + 495)\sigma^4 + (120\epsilon^2 - 135)\sigma^2 + 5 = 0, \tag{3.17}$$

where

$$\Lambda_1 = 64\epsilon^6 - 432\epsilon^4 + 792\epsilon^2 - 429.$$

ϵ	2σ	Re[G]	Im[G]
0.1	0.420614	−5.3742	0.1479
0.1	1.187265	−4.7758	0.4147
0.1	1.745577	−3.5011	0.5923
$\sqrt{3}/2$	0.787075	−7.7669	1.2929
$\sqrt{3}/2$	1.652913	−5.6259	1.5654
$\sqrt{3}/2$	1.925413	−3.5937	0.9903
0.9	0.881537	−8.3335	1.7492
0.9	1.719615	−5.7361	1.7744
0.9	1.942551	−3.6018	1.0309

TABLE 5. Class $\bar{\mathscr{A}}_3$ examples.

It follows that there exist six different inertial oscillation solutions for a given ϵ. The exact solution for σ is given by

$$\sigma_j = \pm\left\{\frac{-1}{3\Lambda_1}(240\epsilon^4 - 720\epsilon^2 + 495) + 2\Gamma\cos\left[\frac{\Phi}{3} + \frac{2(j-1)\pi}{3}\right]\right\}^{1/2} \quad (j = 1, 2, 3), \tag{3.18}$$

where

$$\Gamma = \left[\frac{f_1(\epsilon)}{9\Lambda_1^2}\right]^{1/2}, \quad \Phi = \cos^{-1}\left\{\frac{-1}{\Gamma^3}\left[\frac{f_2(\epsilon)}{54\Lambda_1^3} + \frac{5}{2\Lambda_1}\right]\right\}$$

with f_1 and f_2 are defined as

$$\begin{aligned} f_1(\epsilon) &= 71\,280 - 237\,600\epsilon^2 + 295\,920\epsilon^4 - 164\,160\epsilon^6 + 34\,560\epsilon^8, \\ f_2(\epsilon) &= -15\,436\,575 + 22\,453\,200\epsilon^2 + 57\,736\,800\epsilon^4 - 167\,572\,800\epsilon^6 \\ &\quad + 160\,185\,600\epsilon^8 - 68\,428\,800\epsilon^{10} + 11\,059\,200\epsilon^{12}. \end{aligned}$$

Examples for the frequency 2σ and the corresponding viscous corrections for class $\bar{\mathscr{A}}_3$ are given in table 5 for three different eccentricities. On substitution of (3.18) into the corresponding expressions for the pressure p_0 given by (3.28) at $N = 3$ and the three velocity components given by (3.29)–(3.31) at $N = 3$, we obtain the fully explicit oscillation solutions in closed form for this class.

3.6. *Class* $\bar{\mathscr{S}}_4$

When $N = 4$ with equatorial symmetry (2.43), class $\bar{\mathscr{S}}_4$, the form of the explicit solutions is similar to that for class $\bar{\mathscr{A}}_3$. Again, the expression for $\bar{p}_0$ and the three velocity components (V_η, V_ϕ, V_τ) are contained in the general expressions (3.23)–(3.26) by letting $N = 4$. The normal flow condition that $V_\eta = 0$ at $\eta = \sqrt{1-\epsilon^2}$ yields an equation for σ

$$\Lambda_2\sigma^6 + (336\epsilon^4 - 1232\epsilon^2 + 1001)\sigma^4 + (280\epsilon^2 - 385)\sigma^2 + 35 = 0, \tag{3.19}$$

where

$$\Lambda_2 = 64\epsilon^6 - 528\epsilon^4 + 1144\epsilon^2 - 715.$$

ϵ	2σ	Re[G]	Im[G]
0.1	0.729408	−5.6021	0.2590
0.1	1.358054	−4.8554	0.4781
0.1	1.801232	−3.5124	0.6144
$\sqrt{3}/2$	1.229511	−7.4931	1.7027
$\sqrt{3}/2$	1.757391	−5.4663	1.4298
$\sqrt{3}/2$	1.943665	−3.5728	0.9336
0.9	1.333042	−7.8486	2.1337
0.9	1.807517	−5.5359	1.5738
0.9	1.956750	−3.5779	0.9644

TABLE 6. Class $\bar{\mathscr{S}}_4$ examples.

It follows that there are six axisymmetric oscillation modes for a given ϵ. The exact values of the half oscillation frequency are given by

$$\sigma_j = \pm\left\{\frac{-1}{3\Lambda_2}(336\epsilon^4 - 1232\epsilon^2 + 1001) + 2\Gamma\cos\left[\tfrac{1}{3}\Phi + \tfrac{2}{3}(j-1)\pi\right]\right\}^{1/2} \quad (j = 1, 2, 3), \tag{3.20}$$

where

$$\Gamma = \left[\frac{f_3(\epsilon)}{9\Lambda_2^2}\right]^{1/2}, \quad \Phi = \cos^{-1}\left\{\frac{1}{\Gamma^3}\left[\frac{f_4(\epsilon)}{54\Lambda_2^3} - \frac{35}{2\Lambda_2}\right]\right\}$$

with f_3 and f_4 are defined as

$$\begin{aligned} f_3(\epsilon) &= 176\,176 - 544\,544\epsilon^2 + 619\,696\epsilon^4 - 310\,464\epsilon^6 + 59\,136\epsilon^8, \\ f_4(\epsilon) &= 473\,946\,473 - 1\,416\,983\,568\epsilon^2 + 1\,516\,875\,360\epsilon^4 - 584\,534\,720\epsilon^6 \\ &\quad - 80\,720\,640\epsilon^8 + 114\,250\,752\epsilon^{10} - 21\,676\,032\epsilon^{12}. \end{aligned}$$

Examples for the frequency 2σ and the corresponding viscous corrections for class $\bar{\mathscr{S}}_4$ are shown in table 6 for three different eccentricities. On substitution of (3.20) into the corresponding expressions for the pressure $\bar{p}_0$ given by (3.28) at $N = 4$ and the velocity components given by (3.29)–(3.31) at $N = 4$, we obtain the fully explicit inertial oscillation solution in closed form for this class.

3.7. *Class $\bar{\mathscr{A}}_4$*

Class $\bar{\mathscr{A}}_4$ represents the most complicated explicit solution with symmetry (2.43) which can be written in closed form. By the same method, we can obtain an explicit expression for $\bar{p}_0$ which is then used to derive the three velocity components (V_η, V_ϕ, V_τ) in spheroidal coordinates. The normal flow condition that $V_\eta = 0$ at $\eta = \sqrt{1-\epsilon^2}$ yields an equation for σ

$$C_8^a\sigma^8 + C_6^a\sigma^6 + (2002 - 3080\epsilon^2 + 1120\epsilon^4)\sigma^4 + (-308 + 280\epsilon^2)\sigma^2 + 7 = 0, \tag{3.21}$$

where

$$\begin{aligned} C_8^a &= 2431 - 5720\epsilon^2 + 4576\epsilon^4 - 1408\epsilon^6 + 128\epsilon^8, \\ C_6^a &= -4004 + 8008\epsilon^2 - 4928\epsilon^4 + 896\epsilon^6. \end{aligned}$$

It follows that there exist eight axisymmetric oscillation solutions for a given ϵ, and the analytical expression for σ is too lengthy to be spelled out here. Several examples

ϵ	2σ	Re[G]	Im[G]
0.1	0.332177	−6.1703	0.1193
0.1	0.959559	−5.7577	0.3434
0.1	1.480915	−4.9092	0.5249
0.1	1.840490	−3.5197	0.6304
$\sqrt{3}/2$	0.635585	−8.9976	1.1040
$\sqrt{3}/2$	1.472615	−7.2211	1.7295
$\sqrt{3}/2$	1.819645	−5.3638	1.3168
$\sqrt{3}/2$	1.955839	−3.5600	0.8927
0.9	0.717711	−9.6952	1.5231
0.9	1.560867	−7.4489	2.0526
0.9	1.858459	−5.4115	1.4222
0.9	1.966170	−3.5634	0.9170

TABLE 7. Class $\bar{\mathscr{A}}_4$ examples.

for the frequency 2σ and the corresponding viscous corrections for class $\bar{\mathscr{A}}_4$ are shown in table 7. On substitution of the analytical expression for σ into the pressure $\bar{p}_0$ given by (3.28) at $N = 4$ and the three velocity components given in (3.29)–(3.31) at $N = 4$, we obtain the fully explicit oscillation solutions in closed form for this class.

3.8. *Class $\bar{\mathscr{S}}_5$*

Class $\bar{\mathscr{S}}_5$ represents the spatially most complicated explicit solution with symmetry (2.44) that can still be written in closed form. By the same method, we can obtain an expression for $\bar{p}_0$ and then derive the three velocity components (V_η, V_ϕ, V_τ) in spheroidal polar coordinates. They are contained in the general expressions (3.23)–(3.26) by letting $N = 5$. The half-frequency σ is determined by the vanishing normal flow on the wall of the spheroidal cavity, $V_\eta = 0$ at $\eta = \sqrt{1-\epsilon^2}$, which gives

$$C_8^s\sigma^8 + C_6^s\sigma^6 + (4914 - 6552\epsilon^2 + 2016\epsilon^4)\sigma^4 + (-1092 + 840\epsilon^2)\sigma^2 + 63 = 0, \quad (3.22)$$

where

$$C_8^s = 4199 - 8840\epsilon^2 + 6240\epsilon^4 - 1664\epsilon^6 + 128\epsilon^8,$$
$$C_6^s = -7956 + 14040\epsilon^2 - 7488\epsilon^4 + 1152\epsilon^6.$$

It follows that there exist eight oscillation solutions for a given ϵ. The analytical expression for σ is too lengthy to be shown explicitly. Several examples for the frequency 2σ and the corresponding viscous correction for class $\bar{\mathscr{S}}_5$ are shown in table 8. On substitution of the expression for σ into the corresponding equations for the pressure $\bar{p}_0$ given (3.23) at $N = 5$ and the flow velocity given in (3.24)–(3.26) at $N = 5$, we obtain the fully explicit oscillation solution in closed form for this class.

3.9. *General explicit solutions: classes $\bar{\mathscr{S}}_G$ and $\bar{\mathscr{A}}_G$*

It should be noted that the detailed derivation for the axisymmetric oscillation solutions at different small values of N indicates an essential characteristic for the general explicit solution in spheroidal coordinates valid for any N. On the basis of

ϵ	2σ	Re[G]	Im[G]
0.1	0.594234	−6.4021	0.2149
0.1	1.134337	−5.8683	0.4087
0.1	1.571993	−4.9470	0.5603
0.1	1.869196	−3.5248	0.6424
$\sqrt{3}/2$	1.052919	−8.8275	1.6172
$\sqrt{3}/2$	1.615588	−7.0139	1.6485
$\sqrt{3}/2$	1.859978	−5.2953	1.2276
$\sqrt{3}/2$	1.964395	−3.5518	0.8622
0.9	1.158161	−9.3311	2.0974
0.9	1.687481	−7.1690	1.8872
0.9	1.890890	−5.3298	1.3085
0.9	1.972766	−3.5541	0.8819

TABLE 8. Class $\bar{\mathscr{S}}_5$ examples.

observation, for example,

$$X^2Y^2 = \frac{(\eta\tau)^2\sigma^2}{(1-\epsilon^2\sigma^2)},$$

$$X^2+Y^2 = 1 - \frac{(1-\sigma^2)u^2v^2}{(1-\epsilon^2\sigma^2} + \frac{(\eta\tau)^2\sigma^2}{(1-\epsilon^2\sigma^2},$$

$$X^4+Y^4 = 1 + \frac{(uv)^4(1-\sigma^2)^2}{(1-\epsilon^2\sigma^2)^2} + \frac{(\eta\tau)^4\sigma^4}{(1-\epsilon^2\sigma^2)^2} - \frac{2(uv)^2(1-\sigma^2)}{(1-\epsilon^2\sigma^2)} - \frac{2(uv\tau\eta)^2(1-\sigma^2)\sigma^2}{(1-\epsilon^2\sigma^2)^2},$$

and the pattern suggested, for example, by expression (3.14), we are able to obtain the general expression for $\bar{p}_0$ using explicit spheroidal coordinates in the form

$$\bar{p}_0 = \sum_{i=0}^{N}\sum_{j=0}^{N-i} \bar{C}^s_{ijN}\sigma^{2i}(1-\sigma^2)^j(uv)^{2j}(\eta\tau)^{2i} \tag{3.23}$$

for equatorially symmetric oscillations, where $\bar{C}^s_{ijN}$ is defined by

$$\bar{C}^s_{ijN} = \left[\frac{-1}{(1-\sigma^2\epsilon^2)}\right]^{i+j} \frac{[2(N+i+j)-1]!!}{2^{j+1}(2i-1)!!(N-i-j)!i!(j!)^2},$$

where $N = 2, 3, 4, \ldots$, $(2n-1)!! = (2n-1)(2n-3)\ldots 1$ and $(-1)!! = 1$. The pressure $\bar{p}_0$ represents a general explicit solution for the Poincaré equation given by (2.33). The validity of the general explicit solution can be verified by a direct substitution of (3.23) into (2.33). From (2.26)–(2.29), we can then derive the three velocity components, (V_η, V_ϕ, V_τ), in explicit spheroidal coordinates valid for all possible $N \geqslant 2$

$$V_\eta = -\frac{1}{w}\left[\sum_{i=0}^{N-1}\sum_{j=1}^{N-i} j\bar{C}^s_{ijN}\sigma^{2i+1}(1-\sigma^2)^{j-1}u^{2j-1}v^{2j}\eta^{2i+1}\tau^{2i}\right] + \frac{1}{w}\left[\sum_{i=1}^{N}\sum_{j=0}^{N-i} i\bar{C}^s_{ijN}\sigma^{2i-1}(1-\sigma^2)^j u^{2j+1}v^{2j}\eta^{2i-1}\tau^{2i}\right], \tag{3.24}$$

ϵ	2σ	Re[G]	Im[G]
0.1	0.500928	−7.1093	0.1837
0.1	0.969544	−6.7026	0.3548
0.1	1.376022	−6.0106	0.5011
0.1	1.695101	−4.9953	0.6094
0.1	1.907490	−3.5308	0.6589
$\sqrt{3}/2$	0.915457	−9.9750	1.5002
$\sqrt{3}/2$	1.481672	−8.3691	1.7602
$\sqrt{3}/2$	1.767112	−6.7512	1.4556
$\sqrt{3}/2$	1.907781	−5.2134	1.1018
$\sqrt{3}/2$	1.975374	−3.5429	0.8200
0.9	1.016997	−10.6076	1.9921
0.9	1.569052	−8.6260	2.0844
0.9	1.815542	−6.8351	1.5974
0.9	1.928747	−5.2335	1.1546
0.9	1.981201	−3.5440	0.8340

TABLE 9. Class $\bar{\mathscr{S}}_6$ examples.

$$V_\phi = \sum_{i=0}^{N-1}\sum_{j=1}^{N-i} j\bar{C}^s_{ijN}\sigma^{2i}(1-\sigma^2)^{j-1}(uv)^{2j-1}(\eta\tau)^{2i}, \tag{3.25}$$

$$V_\tau = \frac{1}{w}\left[\sum_{i=0}^{N-1}\sum_{j=1}^{N-i} j\bar{C}^s_{ijN}\sigma^{2i+1}(1-\sigma^2)^{j-1}u^{2j}v^{2j-1}\eta^{2i}\tau^{2i+1}\right] + \frac{1}{w}\left[\sum_{i=1}^{N}\sum_{j=0}^{N-i} i\bar{C}^s_{ijN}\sigma^{2i-1}(1-\sigma^2)^{j}u^{2j}v^{2j+1}\eta^{2i}\tau^{2i-1}\right]. \tag{3.26}$$

Note that there are no oscillation solutions when $N \leqslant 1$ for this equatorial symmetry. The value of half-frequency σ is solutions of the equation resulting from the normal flow condition $V_\eta = 0$ at $\eta = \sqrt{(1-\epsilon^2)}$,

$$0 = \sum_{j=0}^{N-1}(-1)^j \frac{[2(2N-j)]!}{j!(2N-j)![2(N-j)-1]!}[(1-\epsilon^2)\sigma^2]^{(N-j-1)}(1-\sigma^2\epsilon^2)^j. \tag{3.27}$$

For a given ϵ and N, there exist $2(N-1)$ different solutions for σ corresponding to $2(N-1)$ axisymmetric oscillation modes. Evidently, exact solutions of (3.27) for σ in closed form exist only when $N \leqslant 5$; the solutions of (3.27) for $N \geqslant 6$ can be readily computed numerically. Several examples of positive σ for $N = 6$, together with their viscous corrections, are presented in table 9. For equatorially symmetric modes at $N = 6$, there exist 10 axisymmetric oscillation modes for a given ϵ; only the positive values of σ are shown.

By the same procedure, we can also obtain the general explicit solution for equatorially antisymmetric oscillations using spheroidal polar coordinates

$$\bar{p}_0 = \sum_{i=0}^{N}\sum_{j=0}^{N-i} \bar{C}^a_{ijN}\sigma^{2i}(1-\sigma^2)^j(uv)^{2j}(\eta\tau)^{2i+1}, \tag{3.28}$$

where $\bar{C}^a_{ijN}$ is defined by

$$\bar{C}^a_{ijN} = \left[\frac{-1}{(1-\sigma^2\epsilon^2)}\right]^{i+j} \frac{[2(N+i+j)+1]!!}{2^{j+1}(2i+1)!!(N-i-j)!i!(j!)^2}$$

for $N = 1, 2, 3, \ldots$. There are no oscillation solutions for this symmetry when $N = 0$. The three components of the velocity in spheroidal polar coordinates are then derived from (2.26)–(2.29)

$$V_\eta = -\frac{1}{w}\left[\sum_{i=0}^{N-1}\sum_{j=1}^{N-i} j\bar{C}^a_{ijN}\sigma^{2i+1}(1-\sigma^2)^{j-1}u^{2j-1}v^{2j}\eta^{2i+2}\tau^{2i+1}\right] + \frac{1}{2w}\left[\sum_{i=0}^{N}\sum_{j=0}^{N-i}(2i+1)\bar{C}^a_{ijN}\sigma^{2i-1}(1-\sigma^2)^j u^{2j+1}v^{2j}\eta^{2i}\tau^{2i+1}\right], \quad (3.29)$$

$$V_\phi = \sum_{i=0}^{N-1}\sum_{j=1}^{N-i}\bar{C}^a_{ijN} j\sigma^{2i}(1-\sigma^2)^{j-1}(uv)^{2j-1}(\eta\tau)^{2i+1}, \quad (3.30)$$

$$V_\tau = \frac{1}{w}\left[\sum_{i=0}^{N-1}\sum_{j=1}^{N-i} j\bar{C}^a_{ijN}\sigma^{2i+1}(1-\sigma^2)^{j-1}u^{2j}v^{2j-1}\eta^{2i+1}\tau^{2i+2}\right] + \frac{1}{2w}\left[\sum_{i=0}^{N}\sum_{j=0}^{N-i}(2i+1)\bar{C}^a_{ijN}\sigma^{2i-1}(1-\sigma^2)^j u^{2j}v^{2j+1}\eta^{2i+1}\tau^{2i}\right]. \quad (3.31)$$

The half-frequency σ of the equatorially antisymmetric inertial oscillation is solutions of the following equation

$$0 = \sum_{j=0}^{N}(-1)^j \frac{[2(2N-j+1)]!}{j!(2N-j+1)![2(N-j)]!}[(1-\epsilon^2)\sigma^2]^{(N-j)}(1-\sigma^2\epsilon^2)^j. \quad (3.32)$$

Examples for the frequency 2σ and the corresponding viscous correction for equatorially antisymmetric modes at $N = 5$, class $\bar{\mathscr{A}}_5$, are shown in table 10 for three different eccentricities.

For any given ϵ and $N \geqslant 1$, there exist $2N$ solutions for σ giving $2N$ axisymmetric oscillation modes. Exact values for σ in closed form exist only when $N \leqslant 4$. However, solutions of (3.32) when $N \geqslant 5$ have to be computed numerically.

For the first time, we have found a possibly complete set of the axisymmetric oscillation modes in a spheroid of arbitrary eccentricity in explicit spheroidal coordinates, which are

2 modes in class $\bar{\mathscr{A}}_1$; 2 modes in class $\bar{\mathscr{S}}_2$;
4 modes in class $\bar{\mathscr{A}}_2$; 4 modes in class $\bar{\mathscr{S}}_3$;
6 modes in class $\bar{\mathscr{A}}_3$; 6 modes in class $\bar{\mathscr{S}}_4$;
8 modes in class $\bar{\mathscr{A}}_4$; 8 modes in class $\bar{\mathscr{S}}_5$;
$2N$ modes in class $\bar{\mathscr{A}}_G$ for each $N = 5, 6, 7, \ldots$;
$2(N-1)$ modes in class $\bar{\mathscr{S}}_G$ for each $N = 6, 7, 8, \ldots$.

On the basis of the explicit solution, (V_η, V_ϕ, V_τ) in spheroidal coordinates, we can provide a rigorous mathematical proof for the property (1.9) (Appendix A). Moreover, we may expand an arbitrary axisymmetric function in a spheroid of arbitrary eccentricity in terms of the axisymmetric eigenfunctions. It is also the explicit solution in spheroidal coordinates that allows us to proceed readily with the higher-order analysis (the effect of viscosity), which is discussed in § 5 and § 6.

ϵ	2σ	Re[G]	Im[G]
0.1	0.274456	−6.8799	0.1001
0.1	0.802441	−6.5731	0.2921
0.1	1.269563	−5.9494	0.4601
0.1	1.641259	−4.9746	0.5877
0.1	1.890812	−3.5283	0.6516
$\sqrt{3}/2$	0.531547	−10.0636	0.9549
$\sqrt{3}/2$	1.314149	−8.5867	1.7665
$\sqrt{3}/2$	1.706121	−6.8625	1.5488
$\sqrt{3}/2$	1.887753	−5.2476	1.1575
$\sqrt{3}/2$	1.970652	−3.5465	0.8386
0.9	0.603037	−10.8648	1.3322
0.9	1.414111	−8.9417	2.1744
0.9	1.764758	−6.9741	1.7299
0.9	1.912963	−5.2735	1.2220
0.9	1.977577	−3.5481	0.8550

TABLE 10. Class $\bar{\mathscr{A}}_5$ examples.

4. Explicit solutions for inertial waves

When $m \geqslant 1$, the leading-order mainstream problem describes non-dissipative azimuthally travelling waves in a rotating spheroidal cavity. As in the problem of inertial oscillation, we shall take three similar steps to derive a general explicit analytical solution for the pressure p_0 and for the three velocity components of the waves. From (2.34)–(2.38), it can be shown that the equatorially symmetric implicit solution for p_0 can be written as

$$p_0 = (uv)^m \sum_{i=0}^{N} \sum_{j=0}^{N} \left[\frac{(-1)^{i+j}}{i!j!}\right] [X^{2N-2j} Y^{2N-2i}] \times \left[\frac{[2(2N+m-i)]![2(2N+m-j)]!}{(2N+m-i)!(2N+m-j)!(2N-2i)!(2N-2j)!}\right], \quad (4.1)$$

where $N = 1, 2, 3, \ldots, m = 1, 2, 3, \ldots$. An arbitrary normalization is used in (4.1). By a similar analysis, we can also derive the equatorially antisymmetric implicit solution for p_0

$$p_0 = \tau\eta(uv)^m \sum_{i=0}^{N} \sum_{j=0}^{N} \left[\frac{(-1)^{i+j}}{i!j!}\right] [X^{2N-2j} Y^{2N-2i}] \times \left[\frac{[2(2N+m-i+1)]![2(2N+m-j+1)]!}{(2N+m-i+1)!(2N+m-j+1)!(2N-2i+1)!(2N-2j+1)!}\right], \quad (4.2)$$

where $N = 0, 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$.

The explicit inertial wave solution in spheroidal coordinates η and τ can be divided into the following six classes according to its equatorial symmetry and spatial structure:

(i) Class $\mathscr{A}_0$: symmetry (2.44) with $N = 0, m = 1, 2, 3, \ldots$;
(ii) Class $\mathscr{S}_1$: symmetry (2.43) with $N = 1, m = 1, 2, 3, \ldots$;
(iii) Class $\mathscr{A}_1$: symmetry (2.44) with $N = 1, m = 1, 2, 3, \ldots$;
(iv) Class $\mathscr{S}_2$: symmetry (2.43) with $N = 2, m = 1, 2, 3, \ldots$;

m, ϵ	2σ	Re[G]	Im[G]
$1, 0.1$	1.005025	−2.6247	0.2662
$2, 0.1$	0.671141	−3.5260	0.1829
$1, \sqrt{3}/2$	1.600000	−3.0319	1.4284
$2, \sqrt{3}/2$	1.333333	−4.4782	1.7253
$1, 0.9$	1.680672	−3.0752	1.6190
$2, 0.9$	1.449275	−4.6120	2.0787

TABLE 11. Class $\mathscr{A}_0$ examples.

(v) Class $\mathscr{A}_G$: general explicit solution with symmetry (2.44), $N \geqslant 2, m = 1, 2, 3, \ldots$;

(vi) Class $\mathscr{S}_G$: general explicit solution with symmetry (2.43), $N \geqslant 3, m = 1, 2, 3, \ldots$.

For classes (i)–(iv), analytical solutions for the inertial waves in closed form can be obtained; for classes (v)–(vi), exact solutions for the frequency of inertial waves in closed form cannot be obtained.

4.1. *Class* $\mathscr{A}_0$

The simplest equatorially antisymmetric inertial wave, class $\mathscr{A}_0$, is an ideal example for illustration because the solution is simple enough to write out all the mathematical details. The pressure p_0 for class $\mathscr{A}_0$ can be obtained by letting $N = 0$ in (4.2) and using the transformation laws (2.36)–(2.37), which yields

$$p_0 = \frac{(2m+1)!!}{2m!} u^m v^m \eta\tau, \tag{4.3}$$

where p_0 is normalized such that it is consistent with the general solution. With the availability of an explicit expression for p_0 in terms of η and τ, we then derive the velocity of the inertial wave in spheroidal polar coordinates by using (2.26)–(2.29)

$$V_\eta = \frac{(2m+1)!!}{4m!} \frac{\tau(uv)^m}{w} \left[\frac{-\eta^2 m}{u(1-\sigma)} + \frac{u}{\sigma}\right], \tag{4.4}$$

$$V_\phi = \frac{(2m+1)!!}{4m!} \frac{m\eta\tau}{(1-\sigma)} (uv)^{(m-1)}, \tag{4.5}$$

$$V_\tau = \frac{(2m+1)!!}{4m!} \frac{\eta(uv)^m}{w} \left[\frac{\tau^2 m}{v(1-\sigma)} + \frac{v}{\sigma}\right]. \tag{4.6}$$

To determine the half-frequency σ of the inertial wave, we demand the vanishing of the normal velocity V_η, given by (4.4), at the envelope of the spheroidal cavity $\eta = \sqrt{1-\epsilon^2}$. This results in

$$\sigma = \frac{1}{1 + m(1-\epsilon^2)}. \tag{4.7}$$

Several examples for the half frequency σ and the corresponding viscous corrections for equatorially symmetric waves, $\mathscr{A}_0$, are shown in table 11 for three different eccentricities.

It follows that there exists one inertial wave solution for each $m > 0$. Inserting expression (4.7) into (4.4)–(4.6), we obtain the three velocity components of the inertial wave of this class in closed form

$$V_\eta = \frac{(2m+1)!![1+m(1-\epsilon^2)]}{4m!}\frac{\tau(uv)^m}{w}\left[\frac{-\eta^2}{u(1-\epsilon^2)}+u\right], \tag{4.8}$$

$$V_\phi = \frac{(2m+1)!![1+m(1-\epsilon^2)]}{4m!}\eta\tau(uv)^{(m-1)}\frac{1}{(1-\epsilon^2)}, \tag{4.9}$$

$$V_\tau = \frac{(2m+1)!![1+m(1-\epsilon^2)]}{4m!}\frac{\eta(uv)^m}{w}\left[\frac{\tau^2}{v(1-\epsilon^2)}+v\right]. \tag{4.10}$$

The simplest solution in this class is the spin-over mode which is obtained by setting $m = 1$ in (4.8)–(4.10). It is characterized by the vanishing of the radial flow in the spherical limit

$$V_\eta(\eta,\tau)\rightarrow 0 \quad \text{as} \quad \epsilon\rightarrow 0, \tag{4.11}$$

i.e. the wave motion is purely toroidal.

4.2. *Class $\mathscr{S}_1$*

The simplest equatorially symmetric solution can be obtained by letting $N = 1$ in (4.1) together with the transformation laws (2.36)–(2.37), which leads to

$$p_0 = \left[\frac{(2m+1)!!}{2m!}\right](uv)^m - \left[\frac{(2m+3)!!(1-\sigma^2)}{2^2(m+1)!(1-\epsilon^2\sigma^2)}\right](uv)^{(m+2)} - \left[\frac{(2m+3)!!\sigma^2}{2m!(1-\epsilon^2\sigma^2)}\right](uv)^m(\eta\tau)^2. \tag{4.12}$$

By the same procedure as described in class $\mathscr{A}_0$, we can derive the velocity of the inertial wave, which is too lengthy to be spelled out here. It is contained in the general solution (4.21)–(4.23) by setting $N = 1$. The normal flow condition $V_\eta = 0$ at $\eta = \sqrt{1-\epsilon^2}$ gives rise to an equation for σ

$$[(2m+3)(m+2)-2m(m+1)\epsilon^2]\sigma^2 - 2(2m+3)\sigma - m = 0. \tag{4.13}$$

For each non-zero wavenumber m, there exist two inertial wave solutions

$$\sigma = \frac{(2m+3)}{(m+2)(2m+3)-2m(m+1)\epsilon^2}\left\{1\pm\left[\frac{(m+2)^2-1}{(2m+3)}-\frac{2m^2(m+1)\epsilon^2}{(2m+3)^2}\right]^{1/2}\right\}. \tag{4.14}$$

One ($\sigma < 0$) propagates eastward while another ($\sigma > 0$) propagates westward. Several examples for the frequency 2σ and the corresponding viscous corrections are shown in table 12. Class $\mathscr{S}_1$ has the simplest spatial structure for equatorially symmetric inertial waves. In particular, the corresponding fluid motion is largely trapped in the equatorial region with small variation along the direction of rotation axis. In the spherical limit $\epsilon = 0$, these two waves represent the leading-order thermal convection solution for an incompressible viscous fluid with small Prandtl numbers (Zhang 1995).

On substitution of (4.14) into expressions for the velocity (V_η, V_ϕ, V_τ) given by (4.21)–(4.23) at $N = 1$ and for the pressure p_0 (4.12), we obtain the fully explicit solution in closed form for Class $\mathscr{S}_1$.

m, ϵ	2σ	$\mathrm{Re}[G]$	$\mathrm{Im}[G]$
2, 0.1	−0.232083	−4.1930	−0.1547
2, 0.1	+1.236387	−3.5672	+0.4119
4, 0.1	−0.261604	−5.4230	−0.2255
4, 0.1	+0.932335	−5.0125	+0.4047
2, $\sqrt{3}/2$	−0.244987	−6.4832	−0.3594
2, $\sqrt{3}/2$	+1.718671 (1.719)	−3.9365 (−3.936)	+1.4412 (1.441)
4, $\sqrt{3}/2$	−0.293268	−8.2014	−0.5177
4, $\sqrt{3}/2$	+1.515490	−5.9819	+1.9052
2, 0.9	−0.246156	−7.1337	−0.4280
2, 0.9	+1.777885	−3.9727	+1.5898
4, 0.9	−0.296502	−8.9753	−0.6120
4, 0.9	+1.606026	−6.1022	+2.2021

TABLE 12. Class $\mathscr{S}_1$ examples.

4.3. *Class* $\mathscr{A}_1$

When $N = 1$ with equatorial antisymmetry, class $\mathscr{A}_1$, the inertial wave solutions in closed form can still be derived. Equation (4.2) at $N = 1$ together with the transformation laws gives

$$p_0 = \left[\frac{(2m+3)!!}{2m!}\right](\eta\tau)(uv)^m - \left[\frac{(2m+5)!!(1-\sigma^2)}{2^2(m+1)!(1-\epsilon^2\sigma^2)}\right](\eta\tau)(uv)^{(m+2)} - \left[\frac{(2m+5)!!\sigma^2}{6(m)!(1-\epsilon^2\sigma^2)}\right](uv)^m(\eta\tau)^3. \quad (4.15)$$

From (2.26)–(2.29) we then derive the three velocity components (V_η, V_ϕ, V_τ) in spheroidal polar coordinates, which are again too lengthy to be spelled out here and can be obtained by letting $N = 1$ in (4.26)–(4.28). The normal flow condition that $V_\eta = 0$ at $\eta = \sqrt{1-\epsilon^2}$ yields

$$\tfrac{1}{3}A_3\sigma^3 - [(2m+5) - 2(m+2)\epsilon^2]\sigma^2 - [(m+1) - m\epsilon^2]\sigma + 1 = 0, \quad (4.16)$$

where

$$A_3 = (2m+5)(m+3) - (4m^2 + 13m + 12)\epsilon^2 + 2m(m+1)\epsilon^4.$$

It follows that there exist three different inertial waves for each non-zero wavenumber m, which are given by

$$\sigma_j = \frac{(2m+5) - 2(m+2)\epsilon^2}{A_3} + 2\Gamma\cos\left[\frac{\Phi}{3} + \frac{2(j-1)\Pi}{3}\right] \quad (j = 1, 2, 3), \quad (4.17)$$

where

$$\Gamma = \frac{1}{A_3}\left[\sum_{j=0}^{3}\beta_j\epsilon^{2j}\right]^{1/2}, \quad \Phi = \cos^{-1}\left[-\frac{1}{A_3\Gamma^3}\left(\frac{1}{54A_3^2}\sum_{j=0}^{4}\gamma_j\epsilon^{2j} + \frac{3}{2}\right)\right],$$

with

$$\beta_0 = (2m+5)(m^2+6m+8),$$
$$\beta_1 = -(6m^3 + 36m^2 + 76m + 52),$$
$$\beta_2 = 6m^3 + 21m^2 + 30m + 16,$$

m, ϵ	2σ	Re$[G]$	Im$[G]$
2, 0.1	−0.766046	−4.4810	−0.3533
2, 0.1	+0.469758	−4.5766	+0.1398
2, 0.1	+1.500033	−3.5708	+0.5103
4, 0.1	−0.683086	−5.6274	−0.4141
4, 0.1	+0.319757	−5.7140	+0.0960
4, 0.1	+1.224637	−5.0012	+0.5356
2, $\sqrt{3}/2$	−1.203244	−6.1331	−1.7423
2, $\sqrt{3}/2$	+0.963948	−6.4350	+1.5252
2, $\sqrt{3}/2$	+1.839296 (1.839)	−3.7452 (−3.744)	+1.2204 (1.219)
4, $\sqrt{3}/2$	−1.078667	−7.8237	−1.8470
4, $\sqrt{3}/2$	+0.746161	−8.1860	+1.4066
4, $\sqrt{3}/2$	+1.703935	−5.5879	+1.6896
2, 0.9	−1.298689	−6.4685	−2.1864
2, 0.9	+1.079099	−6.8285	+2.0089
2, 0.9	+1.875089	−3.7601	+1.3037
4, 0.9	−1.170572	−8.3037	−2.3497
4, 0.9	+0.856347	−8.7518	+1.9300
4, 0.9	+1.765125	−5.6490	+1.8660

TABLE 13. Class $\mathscr{A}_1$ examples.

$$\begin{aligned}
\beta_3 &= -2m^2(m+1),\\
\gamma_0 &= -27(2m+5)^2(3m^2+16m+19),\\
\gamma_1 &= 81(2m+5)(8m^3+48m^2+98m+64),\\
\gamma_2 &= -(1944m^4+12\,960m^3+33\,129m^2+38\,718m+16\,848),\\
\gamma_3 &= (1296m^4+6264m^3+11\,178m^2+9720m+3456),\\
\gamma_4 &= -324m^2(m+1)(m+2).
\end{aligned}$$

Several examples for the frequency 2σ together with the corresponding viscous corrections for this class, $\mathscr{A}_1$, are shown in table 13. For each non-zero wavenumber m and a given ϵ, there exist three inertial wave solutions. One ($\sigma < 0$) propagates eastward while other two ($\sigma > 0$) propagate westward. On substitution of (4.17) into the corresponding expressions for the pressure p_0 in (4.15) and the flow velocity given by (4.26)–(4.28) at $N = 1$, we obtain the fully explicit inertial wave solution in closed form for this class.

4.4. *Class* $\mathscr{S}_2$

The most complicated class for equatorial symmetric waves that can be written in closed form is $N = 2$, class $\mathscr{S}_2$. By a similar procedure, we can derive the pressure p_0 in spheroidal polar coordinates

$$\begin{aligned}
p_0 = {} & \left[\frac{(2m+3)!!}{4m!}\right](uv)^m - \left[\frac{(2m+5)!!(1-\sigma^2)}{4(m+1)!(1-\sigma^2\epsilon^2)}\right](uv)^{(m+2)}\\
& - \left[\frac{(2m+5)!!\sigma^2}{2m!(1-\sigma^2\epsilon^2)}\right](uv)^m(\eta\tau)^2 + \left[\frac{(2m+7)!!\sigma^2(1-\sigma^2)}{4(m+1)!(1-\sigma^2\epsilon^2)^2}\right](uv)^{(m+2)}(\eta\tau)^2\\
& + \left[\frac{(2m+7)!!(1-\sigma^2)^2}{16(m+2)!(1-\sigma^2\epsilon^2)^2}\right](uv)^{(m+4)} + \left[\frac{(2m+7)!!\sigma^4}{24m!(1-\sigma^2\epsilon^2)^2}\right](uv)^m(\eta\tau)^4. \quad (4.18)
\end{aligned}$$

m, ϵ	2σ	$\mathrm{Re}[G]$	$\mathrm{Im}[G]$
2, 0.1	−1.096086	−4.6308	−0.4773
2, 0.1	−0.101823	−5.1092	−0.0677
2, 0.1	+0.888212	−4.7582	+0.3019
2, 0.1	+1.646272	−3.5685	+0.5640
4, 0.1	−0.968473	−5.7474	−0.5354
4, 0.1	−0.131315	−6.1158	−0.1019
4, 0.1	+0.695534	−5.8680	+0.2690
4, 0.1	+1.408270	−4.9907	+0.6115
2, $\sqrt{3}/2$	−1.562347	−5.6925	−1.7706
2, $\sqrt{3}/2$	−0.104354	−7.7678	−0.1576
2, $\sqrt{3}/2$	+1.439061	−5.9707	+1.7173
2, $\sqrt{3}/2$	+1.894307 (1.894)	−3.6587 (−3.658)	+1.0855 (1.085)
4, $\sqrt{3}/2$	−1.440069	−7.3269	−2.0240
4, $\sqrt{3}/2$	−0.139596	−9.1838	−0.2353
4, $\sqrt{3}/2$	+1.248034	−7.6871	+1.8500
4, $\sqrt{3}/2$	+1.796420	−5.3768	+1.5172
2, 0.9	−1.638539	−5.8499	−2.0523
2, 0.9	−0.104570	−8.5025	−0.1889
2, 0.9	+1.533665	−6.1564	+2.0457
2, 0.9	+1.918457	−3.6658	+1.1401
4, 0.9	−1.526587	−7.5914	−2.4048
4, 0.9	−0.140354	−10.0211	−0.2805
4, 0.9	+1.357649	−8.0046	+2.2944
4, 0.9	+1.840546	−5.4133	+1.6354

TABLE 14. Class $\mathscr{S}_2$ examples.

With the pressure p_0 available, its velocity $\boldsymbol{u}_0$ can be derived ((4.21)–(4.23) at $N = 2$) by using (2.26)–(2.29). The half-frequency σ is determined by the vanishing normal flow on the wall of the spheroidal cavity, $V_\eta = 0$ at $\eta = \sqrt{1-\epsilon^2}$, which gives

$$A_4\sigma^4 + 4(2m+5)\left[(2m+7) - 2(m+2)\epsilon^2\right]\sigma^3 + 6(m+2)\left[(2m+5) - 2m\epsilon^2\right]\sigma^2 - 12(2m+5)\sigma - 3m = 0, \quad (4.19)$$

where

$$A_4 = -(2m+5)(2m+7)(m+4) + 4(2m+5)(m+2)^2\epsilon^2 - 4m(m+1)(m+2)\epsilon^4.$$

Several examples of the frequency 2σ together with the corresponding viscous corrections for class $\mathscr{S}_2$ are shown in table 14.

There exist four different inertial waves for each wavenumber $m > 0$ and any given ϵ. A closed form expression for σ, though complicated and lengthy, can be obtained. Typical spatial structures of the four wave solutions obtained at $\epsilon = 0.1$ and $\epsilon = 0.9$ for $m = 2$ are shown in figures 1 and 2. By inserting the exact solution of (4.19) into the representation of the pressure and velocity given by (4.20)–(4.23) at $N = 2$, we again obtain the fully explicit analytical inertial wave solution in closed form for this class.

It is worth noting that the solution with nearly z-independent structure is always characterized by a slow frequency of the inertial wave (in this case, $\sigma = -0.051$ in figure 1 and $\sigma = -0.052$ in figure 2), leading to an approximately geostrophic balance in (1.3) when $R_o \to 0$ and $E \to 0$.

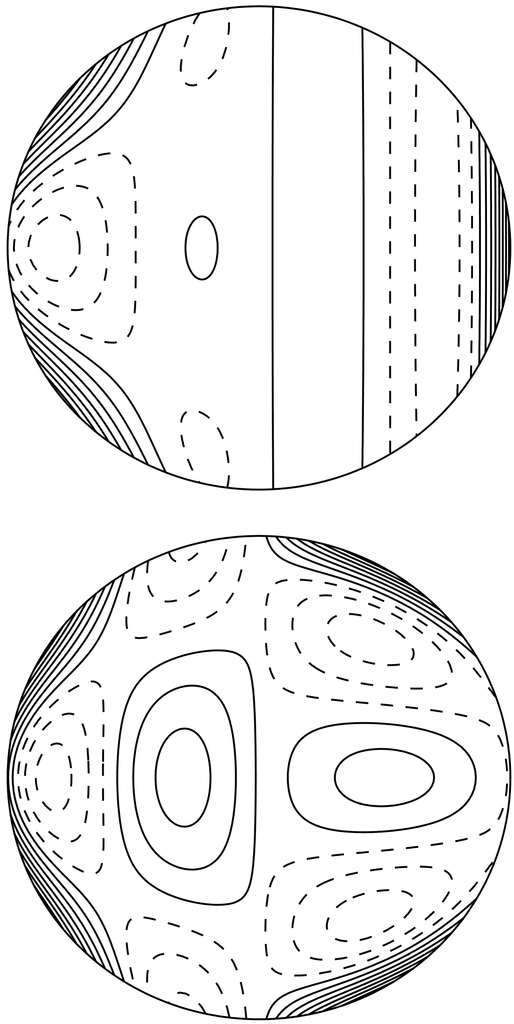

FIGURE 1. Contours of $V_\phi(\eta, \tau)$ in a meridional plane for four different equatorially symmetric inertial wave modes with $N = 2, \epsilon = 0.1$ and $m = 2$. The upper left (right) corresponds to $\sigma = -0.5480(\sigma = -0.0509)$; the lower left (right) is for $\sigma = 0.4441(\sigma = 0.8231)$. The solid contours are for $V_\phi > 0$ while the dashed contours are for $V_\phi < 0$.

4.5. *General explicit solutions: classes $\mathscr{S}_G$ and $\mathscr{A}_G$*

Guided by the key characteristics and patterns of small N solutions, for example (4.18), we are able to obtain the general expression for p_0 in explicit spheroidal coordinates in the form

$$p_0 = \sum_{i=0}^{N} \sum_{j=0}^{N-i} C^s_{ijmN} \sigma^{2i} (1-\sigma^2)^j (uv)^{m+2j} (\eta\tau)^{2i}, \tag{4.20}$$

for equatorially symmetric waves, where C^s_{ijmN} is defined by

$$C^s_{ijmN} = \left[\frac{-1}{(1-\sigma^2\epsilon^2)}\right]^{i+j} \frac{[2(m+N+i+j)-1]!!}{2^{j+1}(2i-1)!!(N-i-j)!i!j!(m+j)!},$$

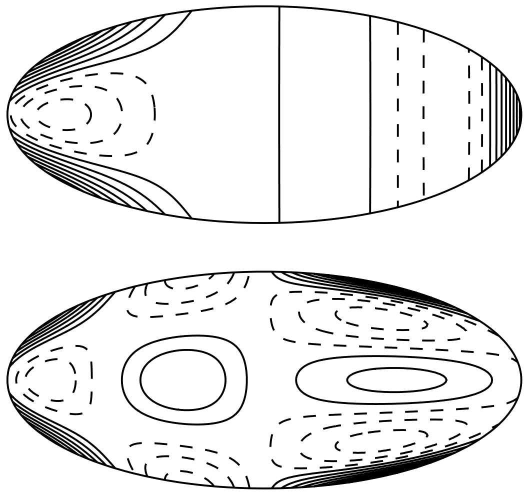

FIGURE 2. Contours of $V_\phi(\eta,\tau)$ in a meridional plane for four different equatorially symmetric inertial waves with $N=2, \epsilon=0.9$ and $m=2$. The upper left (right) corresponds to $\sigma=-0.8193(\sigma=-0.0523)$; the lower left (right) is for $\sigma=0.7668(\sigma=0.9592)$. The solid contours are for $V_\phi>0$ while the dashed contours are for $V_\phi<0$.

where $N=1,2,3,\ldots$. It can be proved by a direct substitution that p_0 given by (4.20) indeed satisfies the relevant Poincaré equation. From (2.26)–(2.29), we can then derive the three velocity components of the general inertial wave, (V_η, V_ϕ, V_τ), in spheroidal polar coordinates valid for all possible N

$$V_\eta = \frac{1}{2w}\sum_{i=0}^{N}\sum_{j=0}^{N-i} C^s_{ijmN}\sigma^{2i-1}(1-\sigma^2)^{j-1}u^{m+2j-1}v^{m+2j}\eta^{2i-1}\tau^{2i} \times [-\eta^2\sigma(2j\sigma+m\sigma+m)+2iu^2(1-\sigma^2)], \quad (4.21)$$

$$V_\phi = \frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{N-i} C^s_{ijmN}\sigma^{2i}(1-\sigma^2)^{j-1}(2j+m+m\sigma)(uv)^{m+2j-1}(\eta\tau)^{2i}, \quad (4.22)$$

$$V_\tau = \frac{1}{2w}\sum_{i=0}^{N}\sum_{j=0}^{N-i} C^s_{ijmN}\sigma^{2i-1}(1-\sigma^2)^{j-1}u^{m+2j}v^{m+2j-1}\eta^{2i}\tau^{2i-1} \times [\tau^2\sigma(2j\sigma+m\sigma+m)+2iv^2(1-\sigma^2)]. \quad (4.23)$$

The value of half-frequency σ is given by solutions of the equation resulting from the normal flow condition $V_\eta=0$ at $\eta=\sqrt{(1-\epsilon^2)}$,

$$0 = m + \sum_{j=0}^{N-1}(-1)^{j+N}\left\{\frac{N![2(2N+m-j)]!(N+m)!}{[2(N-j)]![2(N+m)]!j!(2N+m-j)!}\right\} \times \left[m - \frac{2(1-\sigma)(N-j)}{\sigma(1-\epsilon^2)}\right]\left[\frac{(1-\epsilon^2)\sigma^2}{(1-\sigma^2\epsilon^2)}\right]^{N-j}, \quad (4.24)$$

which is valid for all $N \geqslant 1$ and $m \geqslant 1$. For a given ϵ, m and N, there exist $2N$ different solutions for σ corresponding to $2N$ different inertial waves. Evidently, exact

m, ϵ	2σ	Re[G]	Im[G]
2, 0.1	−1.459112	−4.7728	−0.6168
2, 0.1	−0.808996	−5.6276	−0.3295
2, 0.1	−0.057720	−5.9075	−0.0389
2, 0.1	0.692980	−5.6742	0.2404
2, 0.1	1.339760	−4.9226	0.4738
2, 0.1	1.795596	−3.5622	0.6194
2, 0.9	−1.845236	−5.3356	−1.5948
2, 0.9	−1.394564	−7.7738	−2.2044
2, 0.9	−0.058606	−9.7012	−0.1088
2, 0.9	1.320647	−7.9205	2.1593
2, 0.9	1.805645	−5.5622	1.5872
2, 0.9	1.956472	−3.5890	0.9689

TABLE 15. Class $\mathscr{S}_3$ examples.

solutions of (4.24) for σ in closed form exist only when $N \leqslant 2$; the solutions of (4.24) for $N \geqslant 3$ can be readily computed numerically. For $N = 3$, there are six different inertial waves for each non-zero m and a given ϵ. Several examples for the frequency 2σ together with the corresponding viscous corrections at $N = 3$, class $\mathscr{S}_3$, are shown in table 15.

By the same procedure, we can also obtain the general explicit solution for equatorially antisymmetric waves using spheroidal polar coordinates

$$p_0 = \sum_{i=0}^{N}\sum_{j=0}^{N-i} C^a_{ijmN}\sigma^{2i}(1-\sigma^2)^j(uv)^{m+2j}(\eta\tau)^{2i+1}, \tag{4.25}$$

where C^a_{ijmN} is defined by

$$C^a_{ijmN} = \left[\frac{-1}{(1-\sigma^2\epsilon^2)}\right]^{i+j} \frac{[2(m+N+i+j)+1]!!}{2^{j+1}(2i+1)!!(N-i-j)!i!j!(m+j)!}$$

for $N = 0, 1, 2, \ldots$ and $m = 1, 2, 3, \ldots$. Again its validity can be verified by a direct substitution of (4.25) into the Poincaré equation. The three components of the velocity in spheroidal polar coordinates are then derived from (2.26)–(2.29),

$$V_\eta = \frac{1}{2w}\sum_{i=0}^{N}\sum_{j=0}^{N-i} C^a_{ijmN}\sigma^{2i-1}(1-\sigma^2)^{j-1}u^{m+2j-1}v^{m+2j}\eta^{2i}\tau^{2i+1} \times [-\eta^2\sigma(2j\sigma+m\sigma+m)+(2i+1)u^2(1-\sigma^2)], \tag{4.26}$$

$$V_\phi = \frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{N-i} C^a_{ijmN}\sigma^{2i}(1-\sigma^2)^{j-1}(2j+m+m\sigma)(uv)^{m+2j-1}(\eta\tau)^{2i+1}, \tag{4.27}$$

$$V_\tau = \frac{1}{2w}\sum_{i=0}^{N}\sum_{j=0}^{N-i} C_{ijmN}\sigma^{2i-1}(1-\sigma^2)^{j-1}u^{m+2j}v^{m+2j-1}\eta^{2i+1}\tau^{2i} \times [\tau^2\sigma(2j\sigma+m\sigma+m)+(2i+1)v^2(1-\sigma^2)]. \tag{4.28}$$

m, ϵ	2σ	Re[G]	Im[G]
2, 0.1	−1.311554	−4.7180	−0.5595
2, 0.1	−0.512058	−5.4253	−0.2187
2, 0.1	0.361636	−5.4572	0.1138
2, 0.1	1.157165	−4.8599	0.4043
2, 0.1	1.736217	−3.5653	0.5972
2, 0.9	−1.775895	−5.5215	−1.7926
2, 0.9	−0.973353	−8.3180	−1.9136
2, 0.9	0.845594	−8.4582	1.7484
2, 0.9	1.715407	−5.7783	1.7977
2, 0.9	1.942048	−3.6170	1.0378

TABLE 16. Class $\mathscr{A}_2$ examples.

The half-frequency σ of the equatorially antisymmetric inertial waves is solutions of the following equation

$$0 = \sum_{j=0}^{N} (-1)^j \frac{[2(2N+m-j+1)]!}{[2(N-j)+1]!j!(2N+m-j+1)!} \times \left[m - \frac{(1-\sigma)[2(N-j)+1]}{\sigma(1-\epsilon^2)}\right] \left[\frac{(1-\epsilon^2)\sigma^2}{(1-\sigma^2\epsilon^2)}\right]^{N-j}, \quad (4.29)$$

which is valid for all $N \geqslant 0$ and $m \geqslant 1$. For any given $\epsilon, m > 0$ and N, there exist $2N+1$ different solutions for σ giving $2N+1$ inertial wave modes. Exact solutions of (4.29) for σ in closed form exist only when $N \leqslant 1$; but solutions of (4.29) when $N \geqslant 2$ can be computed numerically. Several examples for the frequency 2σ together with the corresponding viscous corrections at $N = 2$, class $\mathscr{A}_2$, are shown in table 16.

For the first time, we have found a possibly complete set of the inertial wave modes in a spheroid of arbitrary eccentricity in explicit spheroidal coordinates, which are

for each $m > 0$, 1 mode in class $\mathscr{A}_0$;
for each $m > 0$, 2 modes in class $\mathscr{S}_1$;
for each $m > 0$, 3 mode in class $\mathscr{A}_1$;
for each $m > 0$, 4 modes in class $\mathscr{S}_2$;
for each $m > 0$, $2N+1$ modes in class $\bar{\mathscr{A}}_G$ when $N = 2, 3, 4, \ldots$;
for each $m > 0$, $2N$ modes in class $\bar{\mathscr{S}}_G$ when $N = 3, 4, 5, \ldots$.

Based on the explicit general solution of inviscid inertial waves in spheroidal coordinates, we can prove the property (1.9) (Appendix A) and we can derive the higher-order explicit solution taking the viscous effect into account, which is discussed in § 5 and § 6.

5. A general solution for spheroidal Ekman boundary layer

In this section, we derive an explicit general solution for oscillatory Ekman boundary layers in a rotating spheroid of arbitrary eccentricity. The boundary-layer solution is required in solving the $O(E^{1/2})$ interior problem. For a sufficiently small E, we may introduce a stretched variable ξ for the boundary-layer flow in expansion (2.24),

$$\hat{\boldsymbol{u}}_0 = \hat{u}_\tau(\xi,\tau)\boldsymbol{e}_\tau + \hat{u}_\phi(\xi,\tau)\boldsymbol{e}_\phi, \quad \hat{\boldsymbol{u}}_1 = \hat{\boldsymbol{u}}_1(\xi,\tau), \quad \xi = E^{-1/2}[\sqrt{(1-\epsilon^2)} - \eta]. \quad (5.1)$$

On substitution of (2.24) together with (5.1) into (2.16)–(2.18), two differential equations for the leading-order boundary-layer flow, $\hat{\boldsymbol{u}}_0$, can be deduced

$$\left(\mathrm{i}2\sigma\hat{v}^2 - \frac{\partial^2}{\partial\xi^2}\right)^2 \hat{u}_\tau(\xi,\tau) + (2\tau\hat{v})^2\hat{u}_\tau(\xi,\tau) = 0, \tag{5.2}$$

$$\left(\mathrm{i}2\sigma\hat{v}^2 - \frac{\partial^2}{\partial\xi^2}\right)^2 \hat{u}_\phi(\xi,\tau) + (2\tau\hat{v})^2\hat{u}_\phi(\xi,\tau) = 0, \tag{5.3}$$

where $\hat{v} = \sqrt{1 - v^2\epsilon^2}$ is introduced to simplify representation. It is worth noting that the boundary-layer analysis presented in this section is valid for both axisymmetric oscillations ($m = 0$) and inertial waves ($m \geqslant 1$).

The two fourth-order boundary-layer equations are solved subject to the eight boundary conditions, for which the velocity conditions on the spheroidal envelope of the cavity at $\xi = 0$ are

$$\hat{u}_\phi(\xi = 0) = -V_\phi^S(\tau), \tag{5.4}$$

$$\hat{u}_\tau(\xi = 0) = -\mathrm{i}V_\tau^S(\tau), \tag{5.5}$$

$$\frac{\partial^2\hat{u}_\tau}{\partial\xi^2}(\xi = 0) = 2\left[\sigma\hat{v}^2 V_\tau^S(\tau) - \tau\hat{v}V_\phi^S(\tau)\right], \tag{5.6}$$

$$\frac{\partial^2\hat{u}_\phi}{\partial\xi^2}(\xi = 0) = -2\mathrm{i}\left[\sigma\hat{v}^2 V_\phi^S(\tau) - \tau\hat{v}V_\tau^S(\tau)\right], \tag{5.7}$$

where, in the case of equatorially symmetric modes, we have

$$V_\phi^S(\tau) = \frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{N-i} C_{ijmN}^s \sigma^{2i}(1-\sigma^2)^{j-1}(1-\epsilon^2)^i v^{2j+m-1}(2j+m+m\sigma)\tau^{2i}, \tag{5.8}$$

$$V_\tau^S(\tau) = \frac{1}{2\sqrt{1-v^2\epsilon^2}}\sum_{i=0}^{N}\sum_{j=0}^{N-i} C_{ijmN}^s \sigma^{2i-1}(1-\sigma^2)^{j-1}(1-\epsilon^2)^i \times v^{2j+m-1}\tau^{2i-1}[\tau^2\sigma(2j\sigma+m\sigma+m) + 2i(1-\sigma^2)v^2]; \tag{5.9}$$

in the case of equatorially antisymmetric modes, we take

$$V_\phi^S(\tau) = \frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{N-i} C_{ijmN}^a \sigma^{2i}(1-\sigma^2)^{j-1}(1-\epsilon^2)^{i+1/2} v^{2j+m-1}(2j+m+m\sigma)\tau^{2i+1}, \tag{5.10}$$

$$V_\tau^S(\tau) = \frac{1}{2\sqrt{1-v^2\epsilon^2}}\sum_{i=0}^{N}\sum_{j=0}^{N-i} C_{ijmN}^a \sigma^{2i-1}(1-\sigma^2)^{j-1}(1-\epsilon^2)^{i+1/2} \times v^{2j+m-1}\tau^{2i}[\tau^2\sigma(2j\sigma+m\sigma+m) + (2i+1)(1-\sigma^2)v^2]. \tag{5.11}$$

The choice of the boundary conditions (5.8)–(5.11) ensures that the total velocity at the outer spheroidal surface ($\xi = 0$) vanishes. The condition that the boundary-layer flow ($\hat{u}_\eta, \hat{u}_\tau$) must remain bounded gives

$$\hat{u}_\eta(\xi = \infty) = \hat{u}_\phi(\xi = \infty) = 0, \tag{5.12}$$

$$\frac{\partial^2\hat{u}_\tau}{\partial\xi^2}(\xi = \infty) = \frac{\partial^2\hat{u}_\phi}{\partial\xi^2}(\xi = \infty) = 0. \tag{5.13}$$

It is straightforward to derive the solution for the spheroidal Ekman boundary layer satisfying (5.4)–(5.7) and (5.12)–(5.13), which is

$$\hat{u}_\tau(\tau,\xi) = \tfrac{1}{2}\mathrm{i}\big[V_\phi^S(\tau) - V_\tau^S(\tau)\big]\exp\{\alpha_1\xi\} - \tfrac{1}{2}\mathrm{i}\big[V_\phi^S(\tau) + V_\tau^S(\tau)\big]\exp\{\alpha_2\xi\}, \quad (5.14)$$

$$\hat{u}_\phi(\tau,\xi) = -\tfrac{1}{2}\big[V_\phi^S(\tau) - V_\tau^S(\tau)\big]\exp\{\alpha_1\xi\} - \tfrac{1}{2}\big[V_\phi^S(\tau) + V_\tau^S(\tau)\big]\exp\{\alpha_2\xi\}, \quad (5.15)$$

where α_1 and α_2 are defined by

$$\alpha_1^{-1} = -\frac{1}{2|\sigma\hat{v}^2 + \tau\hat{v}|^{1/2}} + \frac{(\sigma\hat{v}^2 + \tau\hat{v})\mathrm{i}}{2|\sigma\hat{v}^2 + \tau\hat{v}|^{3/2}}, \quad (5.16)$$

$$\alpha_2^{-1} = -\frac{1}{2|\sigma\hat{v}^2 - \tau\hat{v}|^{1/2}} + \frac{(\sigma\hat{v}^2 - \tau\hat{v})\mathrm{i}}{2|\sigma\hat{v}^2 - \tau\hat{v}|^{3/2}}, \quad (5.17)$$

An important feature of the Ekman boundary layer is its normal flux which can be determined by inserting (5.1) into (2.19)

$$\frac{\partial}{\partial\xi}[\hat{v}(\boldsymbol{e}_\eta \cdot \hat{\boldsymbol{u}}_1)] = \frac{\partial}{\partial\tau}(v\hat{v}\hat{u}_\tau) + \frac{\mathrm{i}m\hat{v}^2}{v}\hat{u}_\phi. \quad (5.18)$$

A direct integration over ξ leads to the normal flux out of the spheroidal boundary layer

$$(\boldsymbol{e}_\eta \cdot \hat{\boldsymbol{u}}_1)_{\xi=\infty} = \frac{-\mathrm{i}}{2\hat{v}}\frac{\partial}{\partial\tau}\left\{v\hat{v}\left[\frac{\left(V_\phi^S - V_\tau^S\right)}{\alpha_1} - \frac{\left(V_\phi^S + V_\tau^S\right)}{\alpha_2}\right]\right\} + \frac{\mathrm{i}m\hat{v}}{2v}\left[\frac{\left(V_\phi^S - V_\tau^S\right)}{\alpha_1} + \frac{\left(V_\phi^S + V_\tau^S\right)}{\alpha_2}\right], \quad (5.19)$$

which provides the required boundary condition for the $O(E^{1/2})$ interior problem discussed in §6. In deriving (5.19) from (5.18), we should first carry out derivative with respect to τ and then perform integration over ξ.

It is well-known that the boundary-layer equation is broken down at the critical latitudes,

$$\tau_c = \pm\frac{\sigma\sqrt{1-\epsilon^2}}{\sqrt{1-\sigma^2\epsilon^2}}, \quad (5.20)$$

as noticed by Roberts & Stewartson (1965)(Greenspan 1968). Near the critical latitudes $\pm\tau_c$, the spheroidal boundary layer is locally thickened. However, it was suggested by Greenspan (1968) and confirmed by fully numerical solutions in rotating spherical systems (Hollerbach & Kerswell 1995; Zhang 1995) that the breakdown does not significantly affect the value of the viscous decay rate at an asymptotically small E. The value of the decay rate obtained from fully numerical simulations (Hollerbach & Kerswell 1995) agrees to within 1% with the asymptotic decay rate estimated by neglecting the effect of the breakdown (Greenspan 1968; see also Liao *et al.* 2001). An excellent agreement between the fully numerical solution and the asymptotic solution that neglects the local breakdown of the boundary layer was also achieved when the inertial wave is sustained by thermal convection in a rotating sphere (Zhang 1995). We therefore assume that the effect of the local boundary-layer breakdown on the viscous correction can be neglected.

6. Viscous effects on inertial waves and oscillations

The viscous effect takes place at the next-order perturbation, $\boldsymbol{u}_1$ and p_1, in expansions (2.22)–(2.23) for a small but non-zero Ekman number E. It should be noted that the viscous term in (1.3) after using expansion (2.22) may be written as

$$E\nabla^2\boldsymbol{u}_i = E^{1/2}\left[E^{1/2}\nabla^2\boldsymbol{u}_0 + E\nabla^2\boldsymbol{u}_1 + \ldots\right] = O\left(\left|m^2E^{1/2}\boldsymbol{u}_0\right|E^{1/2}\right) + \ldots. \tag{6.1}$$

Equations (2.26)–(2.28) implicitly assume that the maximum wavenumber m of an inertial wave satisfies $m \ll E^{-1/2}$, i.e. the spatial scale of an inertial wave must be much larger than the thickness of the Ekman boundary layer $O(E^{1/2})$ (Kerswell & Barenghi 1995; Greenspan 1968). In the $O(E^{1/2})$ internal problem, we must retain the term $E^{1/2}\nabla^2\boldsymbol{u}_0$ when the scale is given by $m = O(E^{-1/4})$ which gives rise to $E^{1/2}\nabla^2\boldsymbol{u}_0 = O(1)$. The $O(E^{1/2})$ perturbation equations for $\boldsymbol{u}_1$ and p_1 in the general form are

$$2\mathrm{i}\sigma\boldsymbol{u}_1 + 2\boldsymbol{e}_z \times \boldsymbol{u}_1 + \nabla p_1 = E^{1/2}\nabla^2\boldsymbol{u}_0 - G\boldsymbol{u}_0, \tag{6.2}$$

$$\nabla\cdot\boldsymbol{u}_1 = 0. \tag{6.3}$$

When $m \ll O(E^{-1/4})$, the viscous term $E^{1/2}\nabla^2\boldsymbol{u}_0$ in (6.2) may be neglected; it must be retained in (6.2) when $m = O(E^{-1/4})$. In a sense, the term $E^{1/2}\nabla^2\boldsymbol{u}_0$ in (6.2) is associated with the internal viscous dissipation of an inertial wave for the $O(E^{1/2})$ internal problem. However, as we will show, the viscous term in (6.2) makes no contribution in the estimate of the viscous decay rate, a consequence of the unusual properties given by (6.5).

The solvability condition for inhomogeneous differential equation (6.2) is obtained by multiplying it with $\boldsymbol{u}_0^*$, the complex conjugate of $\boldsymbol{u}_0$, and integrating the resulting equation over the spheroidal cavity V,

$$\int_V [2\boldsymbol{u}_1\cdot(\mathrm{i}\sigma\boldsymbol{u}_0^* - \boldsymbol{k}\times\boldsymbol{u}_0^*) + G|\boldsymbol{u}_0|^2]\,\mathrm{d}V = E^{1/2}\int_V \boldsymbol{u}_0^*\cdot\nabla^2\boldsymbol{u}_0\,\mathrm{d}V. \tag{6.4}$$

After making use of (6.3) and the fact that

$$\int_V \boldsymbol{u}_0^*\cdot\nabla^2\boldsymbol{u}_0\,\mathrm{d}V \equiv 0 \tag{6.5}$$

for all values of σ, N and m, which is briefly proved in Appendix A, the resulting solvability condition leads to the viscous decay factor

$$G = -\left[\int_V |\boldsymbol{u}_0|^2\right]^{-1}\int_S p_0^S(\tau)(\boldsymbol{e}_\eta\cdot\boldsymbol{u}_1)\,\mathrm{d}S, \tag{6.6}$$

where $\int_S$ denotes the integration over the surface of the spheroidal container, $(\boldsymbol{e}_\eta\cdot\boldsymbol{u}_1)$ represents the boundary-layer flux given by (5.19), and p_0^S is related to the pressure p_0 evaluated on the envelope of the spheroidal cavity. For equatorially symmetric solutions,

$$p_0^S(\tau) = \sum_{i=0}^{N}\sum_{j=0}^{N-i} C_{ijmN}^{s}\sigma^{2i}(1-\sigma^2)^j(1-\epsilon^2)^i\upsilon^{2j+m}\tau^{2i}; \tag{6.7}$$

for equatorially antisymmetric solutions, we take

$$p_0^S(\tau) = \sum_{i=0}^{N}\sum_{j=0}^{N-i} C_{ijmN}^{a}\sigma^{2i}(1-\sigma^2)^j(1-\epsilon^2)^{i+1/2}\upsilon^{2j+m}\tau^{2i+1}. \tag{6.8}$$

Moreover, the volume integration in (6.6) can be carried out exactly using the general explicit solution for $\boldsymbol{u}_0$. For equatorially symmetric modes, we obtain

$$\begin{aligned}\int_V |\boldsymbol{u}_0|^2\,\mathrm{d}V &= \pi\sum_{i=0}^{N}\sum_{j=0}^{N-i}\sum_{k=0}^{N}\sum_{l=0}^{N-k}(-1)^{i+j+k+l}2^{m+j+l-1}C^s_{ijmN}C^s_{klmN}(1-\epsilon^2)^{i+k+1/2}\\ &\quad\times\left[\frac{(m+j+l-1)!(2i+2k-1)!!\mathscr{Q}}{(1-\sigma^2)^2}+\frac{8ik(2i+2k-3)!!(m+j+l)!}{\sigma^2(1-\epsilon^2)}\right]\\ &\quad\times\left[\frac{\sigma^{2(i+k)}(1-\sigma^2)^{j+l}}{[2(m+j+l+i+k)+1)!!}\right],\end{aligned}\qquad(6.9)$$

where we define that $(-1)!=1$, $(-3)!!=1$ and

$$\mathscr{Q}=(2j\sigma+m\sigma+m)(2l\sigma+m\sigma+m)+(2j+m+m\sigma)(2l+m+m\sigma).$$

For equatorially antisymmetric modes, we have

$$\begin{aligned}\int_V |\boldsymbol{u}_0|^2\,\mathrm{d}V &= \pi\sum_{i=0}^{N}\sum_{j=0}^{N-i}\sum_{k=0}^{N}\sum_{l=0}^{N-k}(-1)^{i+j+k+l}2^{m+j+l-1}C^a_{ijmN}C^a_{klmN}(1-\epsilon^2)^{i+k+3/2}\\ &\quad\times\left[\frac{(m+j+l-1)!(2i+2k+1)\mathscr{Q}}{(1-\sigma^2)^2}+\frac{2(2i+1)(2k+1)(m+j+l)!}{\sigma^2(1-\epsilon^2)}\right]\\ &\quad\times\left[\frac{\sigma^{2(i+k)}(1-\sigma^2)^{j+l}(2i+2k-1))!!}{[2(m+j+l+i+k)+3)!!}\right].\end{aligned}\qquad(6.10)$$

After substitution of (5.19) into the solvability condition (6.6) for the $O(E^{1/2})$ interior problem, we obtain

$$\mathrm{Re}[G]=\pi\left[\int_V|\boldsymbol{u}_0|^2\,\mathrm{d}V\right]^{-1}\int_{-1}^{+1}\left\{\frac{\sqrt{\hat{v}}(\sigma\hat{v}+\tau)(V^S_\phi-V^S_\tau)}{|\sigma\hat{v}+\tau|^{3/2}}\left[v\frac{\mathrm{d}p^S_0}{\mathrm{d}\tau}+\frac{m\hat{v}p^S_0}{v}\right]\right\}\mathrm{d}\tau,\qquad(6.11)$$

$$\mathrm{Im}[G]=\pi\left[\int_V|\boldsymbol{u}_0|^2\,\mathrm{d}V\right]^{-1}\int_{-1}^{+1}\left\{\frac{\sqrt{\hat{v}}\left(V^S_\phi-V^S_\tau\right)}{|\sigma\hat{v}+\tau|^{1/2}}\left[v\frac{\mathrm{d}p^S_0}{\mathrm{d}\tau}+\frac{m\hat{v}p^S_0}{v}\right]\right\}\mathrm{d}\tau.\qquad(6.12)$$

Here, we have used the fact that contribution from the α_1-term in (5.19) is exactly the same as that from the α_2-term because of symmetry with respect to τ. The general expressions (6.11) and (6.12) can be used to calculate the viscous correction for the frequency of any inertial waves ($m\geqslant 1$) or axisymmetric oscillations ($m=0$). The real part of G represents the viscous decay rate of an inertial mode while its imaginary part provides the $O(E^{1/2})$ correction to its frequency. There are four different cases:

(i) *Equatorially symmetric axisymmetric oscillations* ($m=0$). In (6.11) and (6.12), V^S_τ and V^S_ϕ are given by (5.8)–(5.9), p^S_0 by (6.7) and the half-frequency σ by (3.27). The parameter N takes $2, 3, 4, \ldots$ and there are $2(N-1)$ axisymmetric oscillation modes for each N.

(ii) *Equatorially symmetric inertial waves* ($m\geqslant 1$). In (6.11) and (6.12), V^S_τ and V^S_ϕ are given by (5.8)–(5.9), p^S_0 by (6.7) and the half-frequency σ by (4.24). The parameter N takes $1, 2, 3, 4, \ldots$ and there are $2N$ inertial wave modes for each $m\geqslant 1$.

(iii) *Equatorially antisymmetric axisymmetric oscillations* ($m=0$). In (6.11) and (6.12), V^S_τ and V^S_ϕ are given by (5.10)–(5.11), p^S_0 by (6.8) and the half-frequency σ by (3.32). The parameter N takes $1, 2, 3, 4, \ldots$ and there are $2N$ inertial oscillation modes for each N.

(iv) *Equatorially antisymmetric inertial waves* ($m \geqslant 1$). In (6.11) and (6.12), V_τ^S and V_ϕ^S are given by (5.10)–(5.11), p_0^S by (6.8) and the half-frequency σ by (4.29). The parameter N takes $0, 2, 3, 4, \ldots$ and there are $(2N+1)$ inertial wave modes for each $m \geqslant 1$.

The simplest solution describing inertial oscillation in a spheroidal cavity belongs to class $\bar{A}_1$ whose fully explicit solution is given by (3.9)–(3.11). In this case, we have

$$p_0^S(\tau) = \frac{\tau}{2}\sqrt{(1-\epsilon^2)}(5\tau^2 - 3), \tag{6.13}$$

$$V_\phi^S(\tau) = -\frac{3\tau}{4}\frac{(5-4\epsilon^2)}{\sqrt{(1-\epsilon^2)}}\sqrt{(1-\tau^2)}, \tag{6.14}$$

$$V_\tau^S(\tau) = -\frac{3}{4}\frac{\sqrt{1-(1-\tau^2)\epsilon^2}}{\sqrt{(1-\epsilon^2)}}\sqrt{(1-\tau^2)}\sqrt{5-4\epsilon^2}, \tag{6.15}$$

$$\int_V |\boldsymbol{u}_0|^2\,\mathrm{d}V = \frac{3\pi(5-4\epsilon^2)^2}{35\sqrt{(1-\epsilon^2)}}. \tag{6.16}$$

Inserting (6.13)–(6.16) into (6.11) and (6.12), the resulting integration for the viscous correction of the oscillation mode is too complicated and cannot be evaluated exactly for any value of ϵ. However, when ϵ is sufficiently small, we obtain

$$\begin{aligned}
\mathrm{Re}[G] &= -\frac{16}{1375}(155q_1 + 31q_2 + 22q_3 - 22q_4) - 0.7232\epsilon^2 + O(\epsilon^4)\\
&= -3.24774 - 0.7232\epsilon^2 + O(\epsilon^4),\\
\mathrm{Im}[G] &= -\frac{16}{1375}(155q_1 - 31q_2 - 22q_3 - 22q_4) + 0.6110\epsilon^2 + O(\epsilon^4)\\
&= +0.28546 + 0.6110\epsilon^2 + O(\epsilon^4),
\end{aligned}$$

where

$$q_1 = \sqrt{1+\sqrt{5}/5}, q_2 = \sqrt{25-5\sqrt{5}}, \quad q_3 = \sqrt{5-\sqrt{5}}, \quad q_4 = \sqrt{5+\sqrt{5}}.$$

In the limit $\epsilon \to 0$, corresponding to the case of a sphere, we obtain $G = (-3.2477, -0.2855)$. This is in agreement with the numerical values obtained by Kudlick (1966) who gave $G = (-3.248, -0.2853)$.

Several examples of G for different classes of the inertial modes calculated from our general expressions (6.11)–(6.12) are shown in tables 1–16. Kudlick's numerical results are shown in parentheses in the tables. Our results are generally in agreement with those obtained by Kudlick (1966) (see also Greenspan 1968). Evidently, there is a typographical error for the case with $\epsilon = \sqrt{3}/2$ in class $\bar{A}_1$ (Kudlick gave $\mathrm{Im}[G] = 0.1584$, which should be $\mathrm{Im}[G] = 1.584$. Apparently, there are many other errors in his tables. For example, all indices representing the inertial modes in his tables are incorrect). It should be also noted that Kudlick (1966) did not derive or use any explicit analytical solutions for p_0 or (V_η, V_ϕ, V_τ) for any classes of the inertial modes. Consequently, we can only compare our general results with several numerical values given in his three tables.

The simplest example for inertial waves is given by the spin-over mode that is equatorially antisymmetric and $m = 1$ at $N = 0$. By setting $N = 0, m = 1$ in p_0^S, V_τ^S and V_ϕ^S, we obtain

$$\sigma = \frac{1}{2-\epsilon^2}, \quad p_0^S(\tau) = \tfrac{3}{2}\tau\sqrt{(1-\epsilon^2)}\sqrt{(1-\tau^2)}, \tag{6.17}$$

$$V_\phi^S(\tau) = \tfrac{3}{4}\frac{\tau(2-\epsilon^2)}{\sqrt{(1-\epsilon^2)}}, \tag{6.18}$$

$$V_\tau^S(\tau) = \tfrac{3}{4}\frac{(2-\epsilon^2)\sqrt{1-\epsilon^2(1-\tau^2)}}{\sqrt{(1-\epsilon^2)}}, \tag{6.19}$$

$$\int_V |\boldsymbol{u}_0|^2\,\mathrm{d}V = \frac{3\pi}{10}\frac{(2-\epsilon^2)^3}{\sqrt{(1-\epsilon^2)}}. \tag{6.20}$$

Inserting (6.17)–(6.20) into (6.11) and (6.12), the viscous correction for the spin-over mode in a rotating spheroidal cavity of small eccentricity is

$$\begin{aligned}
\mathrm{Re}[G] &= -\frac{3(19+9\sqrt{3})}{28\sqrt{2}} + \frac{(-1039+171\sqrt{3})}{1232\sqrt{2}}\epsilon^2 + O(\epsilon^4) \\
&= -2.62047 - 0.42634\epsilon^2 + O(\epsilon^4), \\
\mathrm{Im}[G] &= -\frac{3(-19+9\sqrt{3})}{28\sqrt{2}} + \frac{(1039+171\sqrt{3})}{1232\sqrt{2}}\epsilon^2 + O(\epsilon^4) \\
&= 0.25846 + 0.76633\epsilon^2 + O(\epsilon^4).
\end{aligned}$$

In the limit $\epsilon \to 0$, corresponding to the case of a sphere, we obtain $\mathrm{Re}[G] = -2.62047$. This is in agreement with that obtained by Greenspan (1968, figure 2.10) who gave $\mathrm{Re}[G] = -2.620$ and is also in agreement with the result of fully numerical simulations by Hollerbach & Kerswell (1995) who obtained $\mathrm{Re}[G] = -2.644$ at $E = 10^{-6.5}$ in a sphere.

7. Summary and remarks

We have obtained the first explicit general asymptotic solution for the axisymmetric oscillations and inertial waves in an incompressible rapidly rotating fluid contained in an oblate spheroidal cavity of arbitrary eccentricity. The general explicit inertial wave solution is comprised of two parts. The first part represents the explicit analytical solution for all the non-dissipative inertial modes in spheroidal polar coordinates whereas the second part is given by the corresponding solution for the spheroidal Ekman boundary layer. The solvability condition at the $O(E^{1/2})$ problem leads to the viscous correction for all the inertial modes.

The explicit spheroidal inertial modes may be used to represent a general time-dependent velocity distribution in spheroidal systems. Consider an inviscid nonlinear flow in a rotating spheroid driven by an external force $\boldsymbol{f}$, which is governed by

$$\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{e}_z \times \boldsymbol{u} + \nabla p = \boldsymbol{f} - R_o \boldsymbol{u}\cdot\nabla\boldsymbol{u}, \tag{7.1}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{7.2}$$

subject to the boundary condition

$$\boldsymbol{e}_\eta\cdot\boldsymbol{u} = 0 \quad \text{at} \quad \eta = \sqrt{1-\epsilon^2}. \tag{7.3}$$

Suppose that both the force $\boldsymbol{f}$ and the flow $\boldsymbol{u}$ are equatorially symmetric. We denote the velocity of an inertial mode in the form

$$\boldsymbol{U}_{Nmn}(\eta,\phi,\tau,\sigma_n) = [\mathrm{i}V_\eta\boldsymbol{e}_\eta + V_\phi\boldsymbol{e}_\phi + \mathrm{i}V_\tau\boldsymbol{e}_\tau]\exp[\mathrm{i}(m\phi)], \tag{7.4}$$

where (V_η, V_ϕ, V_τ) for $m = 0$ are given by (3.24)–(3.26), $\sigma_n, n = 1,\ldots,2N$, can be tabulated using (3.27); (V_η, V_ϕ, V_τ) for $m \geqslant 1$ are given by (4.21)–(4.23),

$\sigma_n, n = 1, \ldots, 2N$, can be tabulated using (4.24). In addition, an inertial mode $\boldsymbol{U}_{Nmj}$ can be normalized such that

$$\int_V |\boldsymbol{U}_{Nmn}|^2 \, \mathrm{d}V = 1. \tag{7.5}$$

Solutions of (7.1)–(7.2) in an oblate spheroid may be expressed as

$$\boldsymbol{u}(\eta, \phi, \tau, t) = \sum_{m,N,n} \left[Z_{Nmn}(t) \boldsymbol{U}_{Nmn} + \text{c.c.} \right], \tag{7.6}$$

where c.c. denotes the complex conjugate of the previous term. The major advantages of the expansion are that (i) the conditions (7.2)–(7.3) are automatically satisfied and (ii) the inertial modes are directly associated with the linear differential operator (the left-hand side) in the equation of motion, (7.1). It follows that solutions of the nonlinear problem in a spheroid can be obtained by solving the ordinary differential equations,

$$\frac{\mathrm{d}Z_{Nmn}}{\mathrm{d}t} = (2i\sigma_n + \mathscr{H}_{Nmn}) \, Z_{Nmn} + R_o \sum_{N_1 m_1 n_1} \sum_{N_2 m_2 n_2} \mathscr{C}_{NN_1N_2} Z_{N_1m_1n_1} Z_{N_2m_2n_2}, \tag{7.7}$$

where

$$\mathscr{H}_{Nmn} = \int_V \boldsymbol{f} \cdot \boldsymbol{U}_{Nmn} \, \mathrm{d}V,$$

and $\mathscr{C}_{NN_1N_2}$ involves the integral of the triple product of the inertial modes. In principle, there are no difficulties in carrying out the integrals in (7.7) by using the explicit solution. It thus offers an effective and straightforward way to solve this difficult nonlinear problem in a rotating spheroid of arbitrary eccentricity. The validity of (7.7), however, is dependent upon the important unanswered mathematical questions (Greenspan 1968, § 2.10) regarding the completeness of the inviscid eigenfunctions and the nature of inviscid eigenvalue spectrum. Although we have obtained the explicit representation for all inertial modes which can be used to expand an arbitrary velocity distribution, the important questions raised by Greenspan in 1968 are still unanswered. We have been unsuccessful in our attempt to prove the completeness of our explicit inertial oscillation and wave solutions. It is a very complicated and difficult task.

Our results also suggest that there is no direct relationship between a single inertial wave given by (4.21)–(4.23) and thermal convection when the Prandtl number of the fluid is moderate or large. Thermal convection in rapidly rotating spherical systems is in the form of a slowly ($\sigma \ll 1$) travelling wave which is nearly independent of the axis of rotation (Roberts 1968; Busse 1970; Fearn 1979; Jones, Soward & Mussa 2000). A slowly travelling and nearly geostrophic wave represents a particular mode of the inertial waves as shown in figure 1. However, the strong phase shift of the convective flow (Zhang 1992) seems to rule out the possibility that the convective flow with moderate Prandtl numbers can be represented by a single inertial wave mode that is slowly travelling and nearly geostrophic.

An intriguing and unusual property of the inertial waves or oscillations is that its dissipation integral vanishes identically in a rotating spheroid of arbitrary eccentricity. Despite the fact that we can prove the property (1.9) rigorously, its physical and mathematical significance is not fully understood. Is there a physical implication in connection with (1.9)? What is the mathematical implication regarding the vanishing of such extremely complicated summations (see Appendix A)? Is there a simpler mathematical proof for (1.9)?

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Appendix A

The following proves that the dissipation integral

$$\int_V \boldsymbol{u}_0 \cdot \nabla^2 \boldsymbol{u}_0 \, \mathrm{d}V \equiv 0$$

for all inertial oscillation and wave solutions, for example, where $\boldsymbol{u}_0$ is given by (3.24)–(3.26) for equatorially symmetric axisymmetric oscillations or by (4.21)–(4.23) for equatorially symmetric inertial waves. It should be mentioned that we (Zhang *et al.* 2001) incorrectly suggested that the above integral vanishes only for a sphere.

We shall present a brief mathematical proof for the case of equatorially symmetric waves because the proof for other cases is almost identical. In light of the explicit solution, it is straightforward to carry out the relevant integrations over the spheroid, but the resulting expression involves rather complex sums with four different indices

$$\int_V \boldsymbol{u}_0 \cdot \nabla^2 \boldsymbol{u}_0 \, \mathrm{d}V = \frac{1}{4\sigma^2(1-\sigma^2)^2}\left[(1+\sigma^2)S_1 + 2m\sigma S_2 + \frac{(1-\sigma^2)^2}{\sigma^2} S_3\right],$$

where

$$S_1 = \frac{\pi 2^m}{\sqrt{(1-\epsilon^2)}} \sum_{i=0}^{N}\sum_{k=1}^{N}\sum_{j=0}^{N-i}\sum_{l=0}^{N-k} (-1)^{i+j+k+l} \frac{\sigma^{2(i+k)}(1-\sigma^2)^{j+l}(1-\epsilon^2)^{i+k}}{(1-\sigma^2\epsilon^2)^{i+j+k+l}}$$
$$\times \frac{[2m(m+l+j)+4jl][2(m+N+i+j)-1]!!}{[2(l+k+i+j+m)-1]!!}$$
$$\times \frac{[2(m+N+k+l)-1]!!}{(2i-1)!!(N-i-j)!i!j!(m+j)!(k-1)!l!} \frac{(2i+2k-3)!!(l+j+m-1)!}{(2k-3)!!(l+m)!(N-k-l)!},$$

$$S_2 = \frac{\pi 2^m}{\sqrt{(1-\epsilon^2)}} \sum_{i=0}^{N}\sum_{k=1}^{N}\sum_{j=0}^{N-i}\sum_{l=0}^{N-k} (-1)^{i+j+k+l} \frac{\sigma^{2(i+k)}(1-\sigma^2)^{j+l}(1-\epsilon^2)^{i+k}}{(1-\sigma^2\epsilon^2)^{i+j+k+l}}$$
$$\times \frac{[2(m+N+i+j)-1]!!}{[2(l+k+i+j+m)-1]!!}$$
$$\times \frac{[2(m+N+k+l)-1]!!}{(2i-1)!!(N-i-j)!i!j!(m+j)!(k-1)!l!} \frac{(2i+2k-3)!!(l+j+m)!}{(2k-3)!!(l+m)!(N-k-l)!},$$

$$S_3 = \frac{4\pi 2^m}{\sqrt{(1-\epsilon^2)^3}} \sum_{i=1}^{N}\sum_{k=2}^{N}\sum_{j=0}^{N-i}\sum_{l=0}^{N-k} (-1)^{i+j+k+l} \frac{\sigma^{2(i+k)}(1-\sigma^2)^{j+l}(1-\epsilon^2)^{i+k}}{(1-\sigma^2\epsilon^2)^{i+j+k+l}}$$
$$\times \frac{[2(m+N+i+j)-1]!!}{[2(l+k+i+j+m)-1]!!}$$
$$\times \frac{[2(m+N+k+l)-1]!!}{(2i-1)!!(N-i-j)!(i-1)!j!(m+j)!(k-2)!l!} \frac{(2i+2k-5)!!(l+j+m)!}{(2k-3)!!(l+m)!(N-k-l)!}.$$

If we can show that

$$S_1 = S_2 = S_3 \equiv 0$$

for all values of ϵ, m, N and σ, then the dissipation integral vanishes identically for all the inertial waves in a rotating fluid spheroid. We present only the brief proof for $S_3 \equiv 0$ since proofs for $S_1 \equiv 0$ and $S_2 \equiv 0$ are similar.

First, we note that the indices, i, j, k, l are intimately entangled and that a direct summation appears to be impossible. An essential step in establishing the result $S_3 = 0$ is to introduce two additional indices, say α and β, by considering a new sum with six indices

$$\begin{aligned}\mathscr{P}_N^M = {} & \frac{4\pi 2^m}{\sqrt{(1-\epsilon^2)^3}} \sum_{\alpha=0}^{M} \sum_{\beta=0}^{M-\alpha} Z_{\alpha,\beta}^M \sum_{i=1}^{N-M} \sum_{k=2}^{N-M} \sum_{j=0}^{N-i-M} \sum_{l=0}^{N-k-M} \\ & \times (-1)^{i+j+k+l} \frac{\sigma^{2(i+k+2\alpha)}(1-\sigma^2)^{j+l+2\beta}(1-\epsilon^2)^{i+k+2\alpha}}{(1-\sigma^2\epsilon^2)^{i+j+k+l+2\alpha+2\beta}} \\ & \times \frac{[2(m+N+i+j+\alpha+\beta)-1]!!}{[2(i+\alpha)-1)!!(N-i-j-M)!(i-1)!j!(m+j+\beta)!} \\ & \times \frac{[2(m+N+k+l+\alpha+\beta)-1]!!}{[2(k+\alpha)-3)!!(N-k-l-M)!(k-2)!l!(m+l+\beta)!} \\ & \times \frac{(m+j+l+\beta)![2(i+k+\alpha)-5)]!!}{[2(m+i+j+k+l+\alpha+\beta+M)-1]!!},\end{aligned}$$

where the coefficients $Z_{i,j}$ are defined as

$$\begin{aligned} Z_{0,0}^0 &= 1; \\ Z_{i,0}^{M+1} &= (-1)^{M+1-i} \frac{(M+1)!}{(M+1-i)!i!} 2^{M+1}; \\ Z_{0,i}^{M+1} &= (-2)^{M+1-i} \frac{(M+1)!}{(M+1-i)!i!}; \\ Z_{i,M+1-i}^{M+1} &= 2^i \frac{(M+1)!}{(M+1-i)!i!}; \\ Z_{i,j}^{M+1} &= -2Z_{i,j}^M + 2Z_{i-1,j}^M + Z_{i,j-1}^M; 1 \leqslant i \leqslant (M-1), 1 \leqslant j \leqslant (M-i). \end{aligned}$$

The precise values of the coefficients $Z_{i,j}^M$ are in fact not required in the proof. Clearly, S_3 and $\mathscr{P}_N^M$ are related by

$$S_3 = \mathscr{P}_N^0.$$

For the purpose of decoupling the entangled indices, we can establish the following recurrence relation after a lengthy analysis,

$$\mathscr{P}_N^0 = \frac{1}{(N-1)} \mathscr{P}_N^1 = \frac{1}{(N-1)(N-2)} \mathscr{P}_N^2 = \cdots = \frac{1}{(N-1)!} \mathscr{P}_N^{N-2}.$$

When $M = N-2$, the indices (i, j) and (k, l) are decoupled such that the relevant summations can be carried out explicitly

$$\begin{aligned}\mathscr{P}_N^{N-2} = {} & \frac{4\pi 2^m}{\sqrt{(1-\epsilon^2)^3}} \sum_{\alpha=0}^{N-2} \sum_{\beta=0}^{N-2-\alpha} Z_{\alpha,\beta}^{N-2} \frac{[2(m+N+\alpha+\beta)+3]!!}{(2\alpha+1)!!(m+\beta)!} \\ & \times \left[\sum_{i=1}^{2} \sum_{j=0}^{2-i} \frac{\sigma^{2i}(1-\sigma^2)^j(1-\epsilon^2)^i}{(1-\sigma^2\epsilon^2)^{i+j}} \frac{(-1)^{i+j}}{(i-1)!j!(2-i-j)!} \right] W_{\alpha\beta}(\sigma, \epsilon),\end{aligned}$$

where $W_{\alpha\beta}(\sigma,\epsilon)$ is a function of σ and ϵ. Here, $\mathscr{P}_N^{N-2}$ is identically zero because

$$\sum_{i=1}^{2}\sum_{j=0}^{2-i}\left[\frac{\sigma^{2i}(1-\sigma^2)^j(1-\epsilon^2)^i}{(1-\sigma^2\epsilon^2)^{i+j}}\right]\left[\frac{(-1)^{i+j}}{(i-1)!\,j!\,(2-i-j)!}\right]\equiv 0.$$

This implies, by the recurrence relation, that $S_3 \equiv 0$.

REFERENCES

Bryan, G. H. 1889 The waves on a rotating liquid spheroid of finite ellipticity. *Phil. Trans. R. Soc. Lond.* A **180**, 187–219.

Busse, F. H. 1970 Thermal instabilities in rapidly rotating systems. *J. Fluid Mech.* **44**, 441–460.

Fearn, D. R. 1979 Thermally driven hydrodynamic convection in a rapidly rotating sphere. *Proc. R. Soc. Lond.* A **369**, 227–242.

Greenspan, H. P. 1964 On the transient motions of a contained rotating fluid. *J. Fluid Mech.* **21**, 673–696.

Greenspan, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.

Gubbins, D. & Roberts, P. H. 1987 Magnetohydrodynamics of the Earth's core. In *Geomagnetism,* vol. 2 (ed. J. A. Jacobs), pp. 1–183. Academic.

Hollerbach, R. & Kerswell, R. R. 1995 Oscillatory internal shear layers in rotating and precessing flows. *J. Fluid Mech.* **298**, 327–339.

Jones, C. A., Soward A. M. & Mussa A. I. 2000 The onset of convection in a rapidly rotating sphere. *J. Fluid Mech.* **405**, 157–179.

Kerswell, R. R. 1993 The instability of precessing flow. *Geophys. Astrophys. Fluid Dyn.* **71**, 105–143.

Kerswell, R. R. 1994 Tidal excitation of hydromagnetic waves and their damping in the Earth. *J. Fluid Mech.* **274**, 219–241.

Kerswell, R. R. & Barenghi C. F. 1995 On the viscous decay-rates of inertial waves in rotating circular cylinder. *J. Fluid Mech.* **298**, 327–339.

Kudlick, M. D. 1966 On transient motions in a contained, rotating fluid. PhD thesis, Math. Dept. MIT, pp. 32–39.

Liao, X., Zhang, K. & Earnshaw, P. 2001 On the viscous damping of inertial oscillation in planetary fluid interiors. *Phys. Earth Planet. Inter.* **128**, 125–136.

Lyttleton, R. A. 1953 *The Stability of Rotating Liquid Masses*. Cambridge University Press.

Malkus, W. V. R. 1967 Hydromagnetic planetary waves. *J. Fluid Mech.* **28**, 793–802.

Poincaré, H. 1885 Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation. *Acta Mathematica* **7**, 259–380.

Rieutord, M. & Valdettaro, L. 1997 Inertial waves in a rotating spherical shell. *J. Fluid Mech.* **341**, 77–99.

Roberts, P. H. 1968 On the thermal instability of a self-gravitating fluid sphere containing heat sources. *Phil. Trans. R. Soc. Lond.* A **263**, 93–117.

Roberts, P. H. & Stewartson, K. 1965 On the motion of a liquid in a spheroidal cavity of a precessing rigid body. II. *Proc. Cam. Phil. Soc.* **61**, 279–288.

Stewartson, K. & Roberts, P. H. 1963 On the motion of a liquid in a spheroidal cavity of a precessing rigid body. *J. Fluid Mech.* **17**, 1–20.

Zhang, K. 1992 Spiralling columnar convection in rapidly rotating spherical fluid shells. *J. Fluid Mech.* **236**, 535–556.

Zhang, K. 1995 On coupling between the Poincaré equation and the heat equation: non-slip boundary condition. *J. Fluid Mech.* **284**, 349–256.

Zhang, K., Earnshaw, P., Liao, X. & Busse, F. H. 2001 On inertial waves in a rotating fluid sphere. *J. Fluid Mech.* **437**, 103–119.